One Quantifier Will Do in Existential Monadic Second-Order Logic over Pictures

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Abstract. We show that every formula of the existential fragment of monadic second-order logic over picture models (i.e., finite, two-dimensional, coloured grids) is equivalent to one with only one existential monadic quantifier.
The corresponding claim is true for the class of word models ([Tho82]) but not for the class of graphs ([Ott95]).
The class of picture models is of particular interest because it has been used to show the strictness of the different (and more popular) hierarchy of quantifier alternation.

1 Introduction

We study monadic second-order logic (MSO) over finite structures. For a given class of structures, one can consider the following two hierarchies:

1. the quantifier alternation hierarchy, where properties are classified wrt. the number of monadic quantifier alternations required for an MSO-sentence;
2. the existential-quantifier depth hierarchy, where properties in the existential fragment of monadic second-order logic (i.e., on the lowest level of the alternation hierarchy) are classified wrt. the number of existential quantifiers required.

Both hierarchies are strict for graphs, i.e., on each level there are graph properties that are not on the previous. The proof of the strictness of the first ([MT97]) goes via another domain, namely the class of (finite, two-dimensional) *pictures*, i.e., arrays over a finite alphabet. The proof of the strictness of the second ([Ott95]) also uses grid-like structures, but different ones.

In contrast to that, for the class of finite strings, both hierarchies collapse, i.e., every MSO-sentence over strings is equivalent to a sentence whose monadic quantifier prefix consists of only one existential quantifier. The proof for the collapse of the second hierarchy can be found in [Tho82].

In the present paper we show (by an adaption of this proof) that the second hierarchy also collapses for the class of pictures, i.e., in a formula of the existential fragment of monadic second-order logic over pictures, the length of the quantifier prefix can be reduced to one.

2 Definitions

For $1 \leq j \leq n$, we denote $\{j, \ldots, n\}$ by $[j, n]$ and $[1, n]$ by $[n]$.

A *picture of size* (m, n) over a given finite alphabet Γ is an $(m\times n)$-matrix over Γ, i.e. a mapping $[m] \times [n] \to \Gamma$. If P is a picture of size (m, n), we call m and n the *height* and *width* of P, denoted by $\underline{P}$ and $|P|$, respectively. If $i \leq i' \leq \underline{P}$ and $j \leq j' \leq |P|$, then we write

$$P([i,i']\times[j,j']) = \begin{pmatrix} P(i,j) & \ldots & P(i,j') \\ \vdots & \ddots & \vdots \\ P(i',j) & \ldots & P(i',j') \end{pmatrix}.$$

By *picture languages* we refer to sets of pictures. The language of all pictures (all picture of size (m, n), or of height m, or of width n, respectively) over Γ is denoted by $\Gamma^{+,+}$ (or $\Gamma^{m,n}$, or $\Gamma^{m,+}$, or $\Gamma^{+,n}$, respectively).

Consider the signature $\sigma_\Gamma = \{S_1, S_2, (Q_a)_{a\in\Gamma}\}$, where the Q_a are unary predicates. To every $m\times n$-picture P over Γ we associate the *picture model* $\underline{P} := ([m] \times [n], S_1^P, S_2^P, (Q_a^P)_a)$, where $S_1^P = \{((i,j),(i+1,j)) \mid (i,j) \in [m-1] \times [n]\}$ and $S_2^P = \{((i,j),(i,j+1)) \mid (i,j) \in [m] \times [n-1]\}$, and $Q_a^P = P^{-1}(a)$ in the set of all positions that carry the letter $a \in \Gamma$. When the picture P is displayed the usual way, the relations S_1^P and S_2^P are the vertical (respectively horizontal) successor relations.

If $P \in \Gamma^{+,+}$, $Q \in \Omega^{+,+}$ are pictures of the same size over two alphabets Γ and Ω, then we denote by $P \otimes Q$ the picture over $\Gamma \times \Omega$ for which $(P \otimes Q)(x) = (P(x), Q(x))$ for every $x \in \mathrm{dom}P$.

The *word model* $\underline{w}$ associated to a nonempty word $w \in \Gamma^+$ is the structure over signature $\varrho_\Gamma := \{S, (Q_a)_{a\in\Gamma}\}$, where $\mathrm{dom}\underline{w} = \{1, \ldots, |w|\}$ is the set of positions of w, and S^w is the successor relation on $\{1, \ldots, |w|\}$, and Q_a^w is the set of those positions of w that carry an a.

2.1 Monadic Second-Order Logic

We use $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \ldots$ as first-order variables and $\boldsymbol{X}, \boldsymbol{Y}, \ldots$ as monadic second-order variables. Formulas of monadic second-order logic (*MSO-formulas*) are built inductively from atomic ones by using (1) boolean connectives $\vee, \neg$, (2) first-order quantifications of the form $\exists \boldsymbol{x}\varphi$, and (3) second-order quantifications of the form $\exists \boldsymbol{X}\varphi$. Atomic formulas either use the relation symbols of the signature or are of the form $\boldsymbol{X}(\boldsymbol{x})$ or $\boldsymbol{x} = \boldsymbol{y}$, saying that position x is in the set X, respectively that x and y are equal.

First-order formulas are *MSO*-formulas in which no second-order quantifier occurs. The existential fragment *EMSO* of *MSO* consists of formulas of the form $\exists \boldsymbol{X}_1 \ldots \exists \boldsymbol{X}_t \varphi$, where φ is first-order.

We write $\varphi(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_t)$ if φ is a formula with free second-order variables among $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_t$. If $X_1, \ldots, X_t \subseteq \mathrm{dom}M$ for a structure M such that φ holds in M under the assignment mapping $\boldsymbol{X}_i$ to X_i, we write $M \models \varphi[X_1, \ldots, X_t]$.

Monadic Second-Order Logic Over Picture and Word Models. For formulas for picture models over a finite alphabet Γ, we use the signature $\sigma_\Gamma := \{S_1, S_2, (Q_a)_{a\in\Gamma}\}$, where S_1 and S_2 are binary and the Q_a are unary. Thus atomic formulas are of the form $S_1(\boldsymbol{x}, \boldsymbol{y})$, $S_2(\boldsymbol{x}, \boldsymbol{y})$, or $Q_a(\boldsymbol{x})$, saying that y is a vertical successor of x, or that y is a horizontal successor of x, or that position x carries the letter a, respectively.

For word models, we use the signature $\varrho_\Gamma := \{S, (Q_a)_{a\in\Gamma}\}$, where S is binary. Now the atomic formula $S(\boldsymbol{x}, \boldsymbol{y})$ says that y is a successor of x.

For a formula φ over σ_Γ (respectively ϱ_Γ) we write $Mod(\varphi)$ for the set of pictures (respectively words) over Γ associated to models of φ.

The *picture language* (over alphabet Γ) *defined by* a sentence φ of signature σ_Γ) is the set of pictures whose associated picture models make φ true. If a picture language is defined by some *EMSO*-sentence over σ_Γ then it is called *EMSO-definable.*

3 Compression of Existential Quantifier Block

3.1 EMSO vs. Locality

A word language $L \subseteq \Gamma^+$ is *local* iff there are sets $A, B \subseteq \Gamma$ and $C \subseteq \Gamma^2$ such that $L = (A\Gamma^* \cap \Gamma^* B) \setminus (\Gamma^* C \Gamma^*)$. The following remark about regular word languages is folklore.

Remark 1. Every string language definable in existential monadic second-order logic is a projection of a local string language.

The above remark also holds if the word "existential" is removed. The following is shown in [Tho82].

Theorem 2. *Let L be a a projection of a local string language. Then there exists a first-order formula $\varphi(\boldsymbol{X})$ in the signature ϱ_Γ such that $L = Mod(\exists \boldsymbol{X}\, \varphi(\boldsymbol{X}))$.*

The proof idea is a follows: Let M be a local word language over alphabet Γ and $L = \pi(M) \subseteq \Sigma^*$ for an alphabet projection $\pi : \Gamma \to \Sigma$. A word $u \in M$ is called a *run* on $\pi(u)$, and letters from Γ are called *states.* A string w over Σ is partioned into sufficiently large sequences such that a $\{0, 1\}$-colouring of such a sequence can encode the first state of the corresponding substring of a run on w. Now the existence of a run on w can be checked by a formula of the required form: φ checks that $\boldsymbol{X}$ corresponds to a $\{0, 1\}$-colouring that encodes the first states of all sequences of a run on w.

The above two results give the following "compression corollary" that says that the the number of existential quantifiers in *EMSO*-formulas can be reduced to one.

Corollary 3. *Every sentence of existential monadic second-order logic over words is equivalent to a sentence of the form $\exists \boldsymbol{X}\, \varphi(\boldsymbol{X})$, where φ is first-order.*

The contribution of this paper is to transfer the above proof and result to pictures languages, i.e., we will show the following analogue of Corollary 3:

Theorem 4. *Every sentence of existential monadic second-order logic over pictures is equivalent to a sentence of the form $\exists \boldsymbol{X} \varphi(\boldsymbol{X})$, where φ is first-order.*

To show this theorem, we proceed in three steps carried out in the next subsections. At the end of this section, we conclude that this theorem holds for formulas, too.

3.2 Domino-Local Picture Languages

The first step is to transfer the notion of locality from words to pictures.

Definition 5. *Let P be a picture, and Δ_1 (and Δ_2) be a set of pictures of size $(2,1)$ (or $(1,2)$, respectively) over the same alphabet. Δ_1 (or Δ_2)* tiles *P iff all subblocks of P of size $(2,1)$ (or $(1,2)$) are in Δ_1 (or Δ_2, respectively)*

Let P be a picture over Γ. The picture that results from P by surrounding it with the fresh boundary symbol $\#$ is denoted by $\hat{P}$.

A picture language L over Γ is domino-local *iff there exist sets $\Delta_1 \subseteq (\Gamma \cup \{\#\})^{2,1}$ and $\Delta_2 \subseteq (\Gamma \cup \{\#\})^{1,2}$ (where $\#$ is a fresh boundary symbol) such that for every picture $P \in L$, both Δ_1 and Δ_2 tile $\hat{P}$. In that case, (Δ_1, Δ_2) is called a* domino tiling system *that* recognizes *L.*

(The notion of "locality" has been introduced in [GRST96] in another way, using 2×2-"tiles" instead of 2×1- and 1×2-tiles. The slightly different and more convenient notion presented here has been studied in [Mat95,LS94]. See [GR96] for a comprehensive survey.)

By *projection* we refer to a mapping from one alphabet to another. A projection is lifted to pictures, words, picture languages, and word languages the obvious way.

Then we have indeed the analogue to Remark 1.

Theorem 6. *([GR96,Mat95,LS94]) Every EMSO-definable picture language is a projection of some domino-local picture language.*

Thus it suffices to show the following in order to obtain Theorem 4:

Theorem 7. *Let L be a projection of a domino-local picture language. Then there exists some first-order formula $\varphi(\boldsymbol{X})$ such that $L = Mod(\exists \boldsymbol{X} \varphi(\boldsymbol{X}))$*

This will be done in the following two subsections.

3.3 Pictures of Bounded Height

The second step is to consider a domino-local picture language restricted to pictures of a fixed height. In this case, the compression of existential quantifier prefixes of EMSO-formulas over picture models can easily be reduced to the word model case.

Theorem 8. *Let L be a projection of a domino-local picture language over Ω and $m \geq 1$ a fixed height. Then there exists an first-order formula $\varphi(\boldsymbol{X})$ (in the signature σ_Ω) such that $Mod(\exists \boldsymbol{X}\, \varphi(\boldsymbol{X}))$ is the set of pictures in L that have height m.*

Proof (Sketch). Let L_0 be the word language over alphabet $\Omega^{m,1}$ that contains all words in $(\Omega^{m,1})^n$ (with $n \geq 1$) that are (as a picture of size (m,n) over Ω) in L.

Application of Theorem 2 to L_0 yields a first-order formula in the signature $\varrho_{\Omega^{m,1}}$, which can be translated to a first-order formula φ in the signature σ_Ω in a straightforward way.

3.4 Pictures of Unbounded Height

The third step is to consider projections of domino-local picture languages without the restriction of a fixed height. We will sketch the main construction.

The aim is to construct a formula of the existential fragment of monadic second-order logic whose models are exactly the pictures of some projection of a given domino-local picture language $M \subseteq \Omega^{+,+}$ under an alphabet projection π. Let (Δ_1, Δ_2) be a domino tiling system recognizing M.

Without the additional limitation to one single monadic quantifier, such a formula may, informally speaking, work as follows: (1) Guess an Ω-colouring of the input picture, and then (2) check that the local restrictions are fulfilled, i.e., the picture obtained this way is tiled by Δ_1 and Δ_2.

Since we are restricted to one single monadic quantifier, the formula is not able to guess the Ω-colouring in step (1) but only a $\{0,1\}$-colouring. However, if we partition the picture into sufficiently large blocks, then it is possible to guess the Ω-colouring *of the border* of each block and store this in a $\{0,1\}$-colouring of the complete block. Then a finite disjunction may check if there really is a Ω-colouring of the block with that border such that Δ_1 and Δ_2 tile the inside of it. What remains to be done is to check whether the colouring of the borders of neighboured blocks fit to each other in the sense that the 2×1 or 1×2-subblocks (of the Ω-coloured picture) along the edges of blocks are in Δ_1 or, respectively, Δ_2. This can be checked (in the actual $\{0,1\}$-coloured picture) by a first-order formula because the information about the Ω-colouring of the border of the block is coded in the $\{0,1\}$-colouring of the inside.

We prepare the proof with some definitions.

Definition 9. *Let $P \in \Gamma^{+,+}$ be a picture of size (m,n). Then*

$$\begin{aligned}
\mathit{left}(P) &= (P(1,1) \ldots P(m,1))^\top \\
\mathit{right}(P) &= (P(1,n) \ldots P(m,n))^\top \\
\mathit{top}(P) &= (P(1,1) \ldots P(1,n)) \\
\mathit{bottom}(P) &= (P(m,1) \ldots P(m,n)) \\
\mathit{border}(P) &= (\mathit{left}(P), \mathit{right}(P), \mathit{top}(P), \mathit{bottom}(P))
\end{aligned}$$

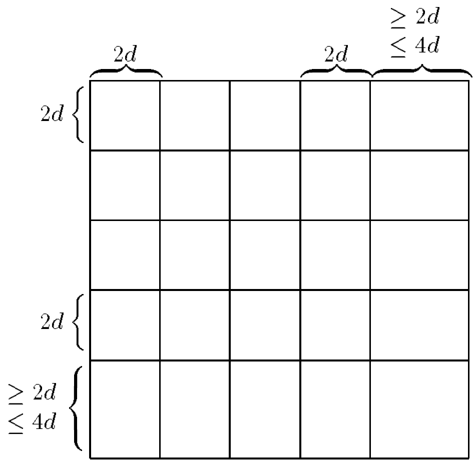

Fig. 1. Partition into Blocks

The next definition will help to define the partition of a picture into blocks.

Definition 10. *Let* $m \geq d \geq 1$*. Choose* $n \geq 0$ *and* $r < d$ *in such a way that* $m = (n+1)d + r$*. The tuple* $(1, d+1, 2d+1, \ldots, nd+1, m+1)$ *is called the* d-step sequence *in* m*.*

Note that if $(i_0, \ldots, i_{n+1})$ is the d-step sequence in m, then $d \leq i_{n+1} - i_n < 2d$.

Let P be some picture of size (m, n) (with $m, n \geq d$). Let i, i' (respectively j, j') be consecutive components in the d-step sequence in m (respectively n). The *d-block of P at position* (i, j) is the subblock $\mathit{Block}(P, d, (i, j)) = P([i, i'-1] \times [j, j'-1])$ of P.

If i'' (respectively j'') are components following i' (respectively j') in these sequences, then $P[i', i''-1] \times [j, j'-1]$ (respectively $P[i, i'-1] \times [j', j''-1]$) will be called a horizontally (respectively vertically) following d-block of P.

Figure 1 illustrates how a picture of size $\geq (2d, 2d)$ is split into $2d$-blocks.

We will make use of the fact that every picture P over Ω whose width and height are $\geq 2d$ can be split into $2d$-blocks, and there are only singly exponentially many essentially different (wrt. tilability by (Δ_1, Δ_2)) types of $2d$-blocks.

Proof. (of Theorem 7.) There is a domino-local picture language M over alphabet $\Omega = \Theta \times \Gamma$ such that L is over alphabet Γ, and L is the image of M under the alphabet projection $\pi : \Omega \to \Gamma$, $(a, b) \mapsto b$. Let (Δ_1, Δ_2) be a domino tiling system that recognizes M.

Choose d such that there exists an injective mapping

$$f : \bigcup_{2d \leq m,n < 4d} (\Omega^m \times \Omega^m \times \Omega^n \times \Omega^n) \to \{0,1\}^{d \times d}.$$

We will only show that there exists a first-order formula $\psi(\boldsymbol{X})$ such that for every picture P over Γ we have $P \in L$ iff $P \models \exists \boldsymbol{X}\, \psi(\boldsymbol{X})$ and size$P \geq (2d, 2d)$.

The claim will then follow from Theorem 8 because for every $m < 2d$, there are formulas $\varrho_m(\boldsymbol{X})$ and $\gamma_m(\boldsymbol{X})$ such that $Mod(\exists \boldsymbol{X} \varrho(\boldsymbol{X})) = L \cap \Gamma^{m,+}$ and $Mod(\exists \boldsymbol{X} \gamma(\boldsymbol{X})) = L \cap \Gamma^{+,m}$, and hence for $\varphi = \psi \vee \bigvee_{m<2d}(\varrho_m \vee \gamma_m)$ we have

$$\begin{aligned} L &= (L \cap \Gamma^{\geq 2d, \geq 2d}) \cup \bigcup_{m<2d} (L \cap \Gamma^{m,+}) \cup (L \cap \Gamma^{+,m}) \\ &= Mod(\exists \boldsymbol{X} \psi \vee \bigvee_{m<2d} (\exists \boldsymbol{X} \varrho_m \vee \exists \boldsymbol{X} \gamma_m) \\ &= Mod(\exists \boldsymbol{X} \varphi). \end{aligned}$$

We proceed with the construction of ψ. For a block B over Ω whose width and height are $\geq 2d$ and $< 4d$, let $ess(B)$ be the picture of same size as B such that

$$ess(B)(i,j) = \begin{cases} 1 & \text{if } i = 0 \vee j = 0 \\ f(border(B))(\frac{1}{2}(i+1), \frac{1}{2}(j+1)) & \text{if } i \equiv_2 j \equiv_2 1 \wedge i, j < 2d \\ 0 & \text{else.} \end{cases}$$

Intuitively, $ess(B)$ carries all the essential information of a block B.

The following are equivalent for every picture P over Γ:

1. $P \in L$
2. There exists a picture Q over Θ of the same size such that Δ_1 and Δ_2 tile $(\widehat{Q \otimes P})$
3. There exists a picture Q over Θ of the same size such that for every $2d$-block B of $(Q \otimes P)$ we have:
   (a) for the horizontally next $2d$-block B_1 of $(Q \otimes P)$ (if present), Δ_2 tiles $(right(B) \ominus left(B_1))^{\top}$;
   (b) for the vertically next $2d$-block B_1 of $(Q \otimes P)$ (if present), Δ_1 tiles $bottom(B) \ominus top(B_1)$;
   (c) if B is a top-most $2d$-block, then Δ_1 tiles $\#^{|B|} \ominus top(B)$;
   (d) if B is a bottom-most $2d$-block, then Δ_1 tiles $bottom(B) \ominus \#^{|B|}$;
   (e) if B is a leftmost $2d$-block, then Δ_2 tiles $(\#^{\underline{\overline{B}}} \ominus left(B))^{\top}$;
   (f) if B is a rightmost $2d$-block, then Δ_2 tiles $(right(B) \ominus \#^{\underline{\overline{B}}})^{\top}$;
   (g) Δ_1 and Δ_2 tile B.
4. There exists a picture Q' over $\{0,1\}$ of the same size such that: for every i from the $2d$-step sequence of $\underline{\overline{P}}$ and every j from the $2d$-step sequence of $|P|$, if $B' = Block(Q', 2d, (i,j))$, then
   (a) there is some $B \in ess^{-1}(B') \cap \pi^{-1}(Block(P, 2d, (i,j)))$ that is tiled by Δ_1 and Δ_2,
   (b) some (and hence any) $2d$-block $B \in ess^{-1}(B')$ satisfies 3a to 3f.

The equivalence of 3 an 4 is due to the fact that properties 3a to 3f only depend on $border(B)$ and hence only on $ess(B)$.

The properties 4b and 4a can be checked by a first-order formula $\psi(\boldsymbol{X})$, where $\boldsymbol{X}$ encodes the $\{0,1\}$-picture Q'. To see the latter we observe that every picture Q' all of whose $2d$-blocks have property 4a has the property that two horizontally (respectively vertically) consecutive 1-positions mark a row (respectively column) whose index is in the $2d$-step sequence of the height (respectively width) of a picture. And for a picture that has this property, it is possible to determine consecutive $2d$-blocks by first-order formulas and hence to check 4b by finite disjunctions ranging over all possible $2d$-blocks over $\{0,1\}$.

This completes the proof of Theorem 7.

From Theorems 6 and 7 we can conclude Theorem 4.

3.5 Compression in the Presence of Free Variables

We wish to show the fact that the "compression" of an existential quantifier block also works for formulas.

Definition 11. *For $m, n \geq 1$ and a tuple $\overline{X} = (X_1, \ldots, X_t)$ of subsets of $[m] \times [n]$ we define the* characteristic $[m] \times [n]$-picture $c_{[m]\times[n]}(\overline{X})$ of $\overline{X}$ *as follows:* $c_{[m]\times[n]}(\overline{X}) : [m] \times [n] \to \{0,1\}^t$, $c_{[m]\times[n]}(\overline{X})(x)(i) = \begin{cases} 1 & \text{if } x \in X_i \\ 0 & \text{else.} \end{cases}$ *for every $x \in [m] \times [n]$ and every $i \leq t$.*

A formula $\varphi(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_t, \boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_r)$ in signature σ_Γ is called equivalent *to a formula $\varphi'(\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_r)$ in $\sigma_{\Gamma\times\{0,1\}^t}$ iff for every $P \in \Gamma^{+,+}$ and every tuple $\overline{X} = (X_1, \ldots, X_t)$ and $\overline{Y} = (Y_1, \ldots, Y_r)$ of subsets of* $\mathrm{dom} P$,

$$\underline{P} \models \varphi[\overline{X}, \overline{Y}] \iff \underline{P \otimes c_{\mathrm{dom}P}(\overline{X})} \models \varphi'[\overline{Y}].$$

Lemma 12. *For every first-order formula $\varphi(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_t)$ in the signature σ_Γ, there is an equivalent first-order formula in the signature $\sigma_{\Gamma\times\{0,1\}^t}$ and vice versa.*

Corollary 13. *Every formula $\exists \overline{\boldsymbol{Y}} \varphi$ in the signature σ_Γ, where $\overline{\boldsymbol{Y}}$ is some variable tuple and φ is first-order, is equivalent to a formula $\exists \boldsymbol{Y}\, \psi$ for some first-order formula ψ in the signature σ_Γ.*

Proof. Let $\overline{\boldsymbol{X}} = (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_t)$, where $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_t$ are the free variables of $\exists \overline{\boldsymbol{Y}} \varphi$. By Lemma 12, $\varphi(\overline{\boldsymbol{X}}, \overline{\boldsymbol{Y}})$ is equivalent to a formula $\varphi'(\overline{\boldsymbol{Y}})$ in the signature $\sigma_{\Gamma\times\{0,1\}^t}$. By Theorem 4, $\exists \overline{\boldsymbol{Y}} \varphi'(\overline{Y})$ is equivalent to some sentence $\exists \boldsymbol{Y}\, \psi'(\boldsymbol{Y})$ for a first-order formula ψ' in signature $\sigma_{\Gamma\times\{0,1\}^t}$. Again using Lemma 12, we obtain a first-order σ_Γ-formula ψ that is equivalent to ψ'.

Then $\exists \boldsymbol{Y}\, \psi$ is equivalent to $\exists \overline{\boldsymbol{Y}} \varphi$.

4 Concluding Remarks

The proof that the number of quantifiers in formulas of existential monadic second-order logic over words can be limited to one has been transfered to pictures, which are the 2-dimensional analogue to words. It is quite obvious that this proof can be transfered to arbitrary finite dimensions by induction.[1]

[1] The crucial point is to reprove Theorem 6 for higher dimensions, which has not been done yet though it is straightforward.

For the four classes of models mentioned in the introduction and the two monadic hierarchies of quantifier alternation and existential quantifier depth, the situation looks as follows. Here, "collapse" always means "collapses to the existential fragment EMSO".

	Quantifier Alternation	Existential Quantifier Depth
Word Models	collapse	collapse (see [Tho82])
Picture Models	strict (see [MT97])	collapse (this paper)
Otto-Grids	(open)	strict (see [Ott95])
Graphs	strict	strict

The results of the bottom row are infered by the results of the second and third row, respectively, by encoding techniques. I conjecture that for the class of "Otto-grids" (as defined in [Ott95]), the quantifier alternation hierarchy collapses.

The class of pictures is of special interest because it has a strict quantifier alternation hierarchy. Regarding this hierarchy, the following question is natural: Can the number of quantifiers in each block of a formula of minimal quantifier alternation be limited to one?

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