GAMES OF LENGTH $\omega \cdot 2$

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Abstract. This note combines an unpublished theorem of Woodin's about AD and Uniformisation with combinatorial arguments of Blass' to get a startling consequence for games on ω of length $\omega \cdot 2$: The determinacy of these games is equivalent to the Axiom of Real Determinacy.

This short note does not contain the proof of an independent new theorem. Instead, it provides a surprising viewpoint on two known theorems by pointing out that they can be used to fill a gap in the theory of determinacy for games of fixed countable length. This remark is particularly puzzling since the conjectures and guesses of the experts about how to fill this gap were anticipating a different solution, and the result seems to defy intuition.

The result of this note has been independently noticed by Richard Ketchersid (Los Angeles).

Throughout this note, we shall work in $\mathsf{ZF} + \mathsf{DC}$. We denote by $\mathsf{AD}[\alpha]$ the Axiom of Determinacy for games of length α with moves in ω, *i.e.*, "for all $A \subseteq \omega^\alpha$ there is either a winning strategy for player I or a winning strategy for player II in the game $G(A)$". Moreover, $\mathsf{AC}_{\mathbb{R}}(\mathbb{R})$ is the Axiom of Choice for families of sets of reals parametrized by real numbers, and $\mathsf{AD}_{\mathbb{R}}$ is the Axiom of Real Determinacy.

The following picture depicts what has been known about the axioms $\mathsf{AD}[\alpha]$:

The axioms $\mathsf{AD}[n]$ for a natural number n are consequences of ZF alone [Zer13], the axioms $\mathsf{AD}[\alpha]$ for $\omega \leq \alpha < \omega \cdot 2$ are equivalent to $\mathsf{ZF} + \mathsf{AD}$ (by a folklore argument mentioned in [Bl75, Remark 7]), the axioms $\mathsf{AD}[\alpha]$ for $\omega^2 \leq \alpha < \omega_1$ are equivalent to $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ (for $\omega^2 \leq \alpha < \omega^3$ this is the main theorem of [Bl75], for $\omega^3 \leq \alpha < \omega_1$ this has been independently shown by Donald A. Martin and Hugh Woodin [unpublished]), and the axioms AD_α

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for uncountable α are inconsistent by a theorem of Mycielski [My63] and Blass [Bl75, Remark 8].

This picture leaves a conspicuous gap between $\omega \cdot 2$ and ω^2. Since games of length ω^2 seem to be the natural choice for coding information about games with real number moves, several people have thought that the determinacy axioms from the gap below ω^2 should strictly weaker than $\mathsf{AD}_{\mathbb{R}}$.

Using an observation of Blass' and an unpublished theorem of Woodin's, we shall show that these conjectures were wrong:

Fact 1 (Blass). $\mathsf{AD}[\omega \cdot 2]$ implies $\mathsf{AC}_{\mathbb{R}}(\mathbb{R})$ [Bl75, Remark 7].

Fact 1 shows that $\mathsf{AD}[\omega \cdot 2]$ cannot be equivalent to AD since under the assumption of AD, $\mathsf{AC}_{\mathbb{R}}(\mathbb{R})$ doesn't hold in $\mathbf{L}(\mathbb{R})$. But much more is true:

Theorem 2 (Woodin). Then $\mathsf{AD} + \mathsf{AC}_{\mathbb{R}}(\mathbb{R})$ and $\mathsf{AD}_{\mathbb{R}}$ are equivalent. (For a reference without proof, *cf.* [Ka94, Theorem 32.23].)

These two results give:

Theorem 3. Suppose $\omega \cdot 2 \leq \alpha < \omega_1$. Then $\mathsf{AD}[\alpha]$ is equivalent to $\mathsf{AD}_{\mathbb{R}}$.

Proof: That $\mathsf{AD}_{\mathbb{R}}$ implies $\mathsf{AD}[\alpha]$ has been known before (by the mentioned theorems of Blass, Martin and Woodin). But the other direction follows from the concatenation of Fact 1 and Theorem 2. q.e.d.

This completes the picture of the axioms $\mathsf{AD}[\alpha]$ in a surprising way as follows:

The question whether you can eliminate the use of DC from the proof of Theorem 2 is an open problem. Consequently, we do not know how to fill the gap without working in the theory $\mathsf{ZF} + \mathsf{DC}$. To try and do so would be a worthwhile project.

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