Planar Graphs: Logical Complexity and Parallel Isomorphism Tests

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Problem: Planar Graph Isomorphism

Input: G and H, two planar graphs

Question: $G \cong H$?

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in AC^1, by Miller-Reif 91 (with use of

the AC^1 embedding algorithm by Ramachandran-Reif 94)

Present result

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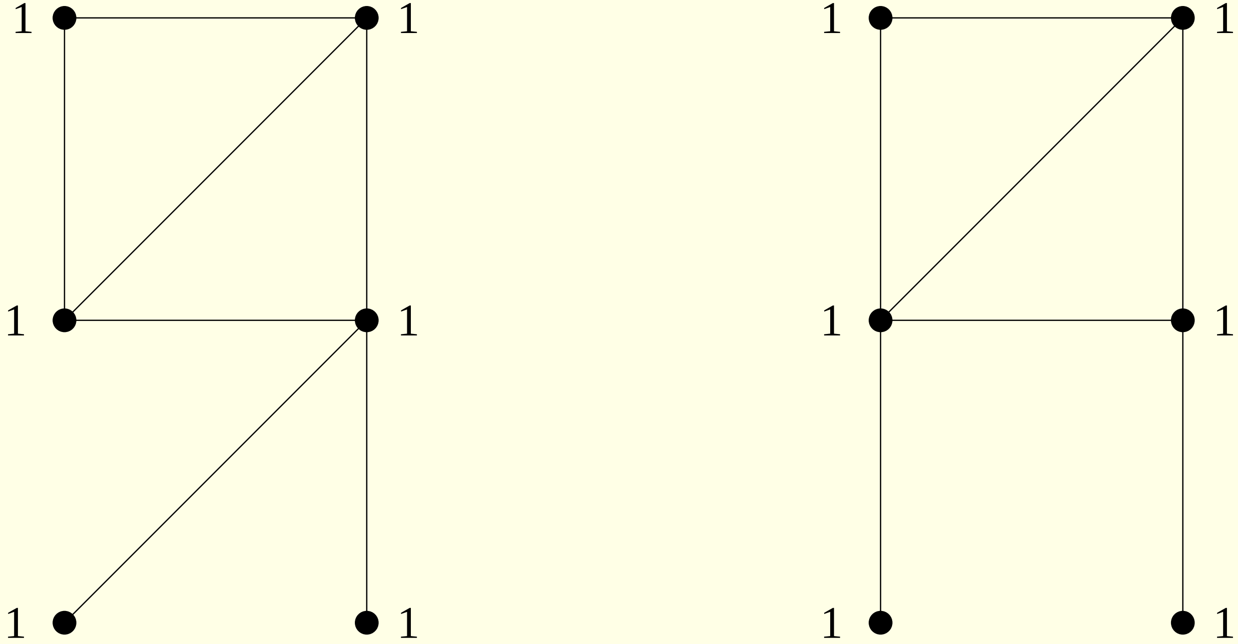
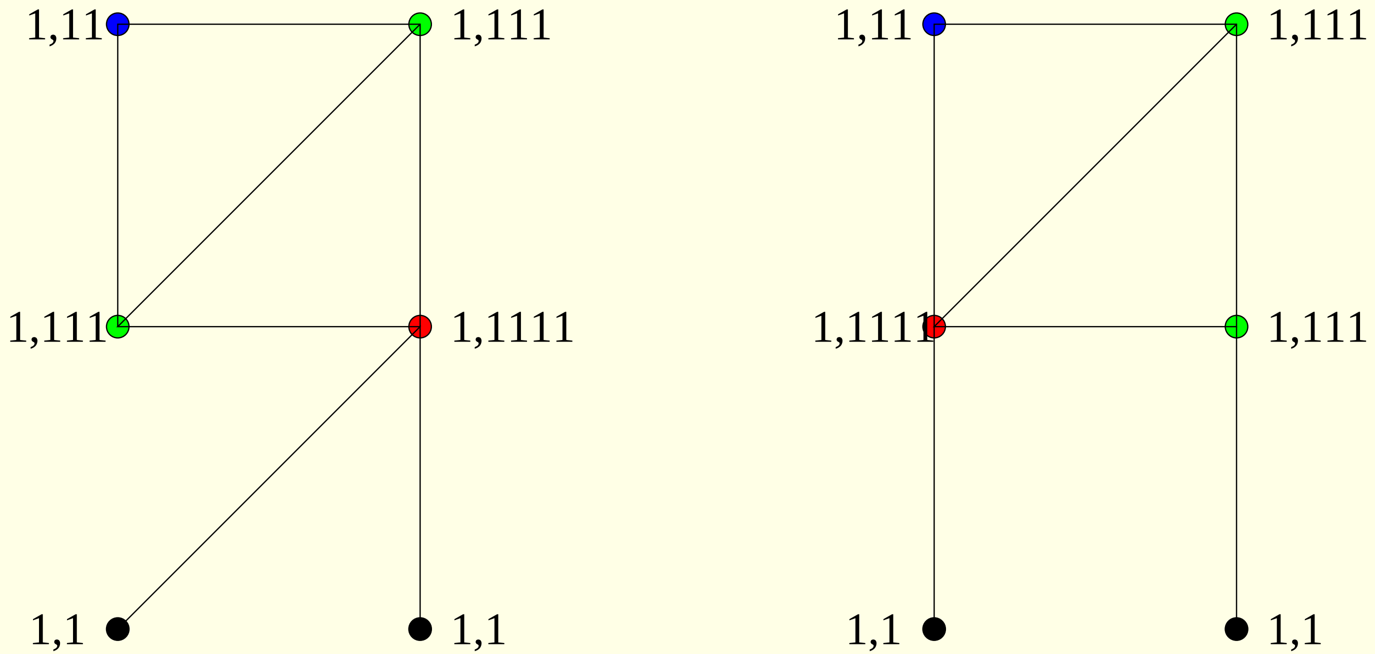
Remark (Miller-Reif): Planar Graph Isomorphism AC^1-reduces to the 3-connected case.

1-dim WL $=$ color refinement procedure



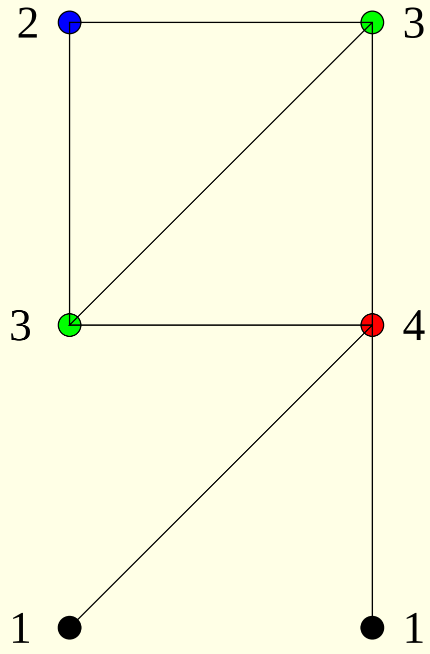
Initial coloring

1-dim WL = color refinement procedure



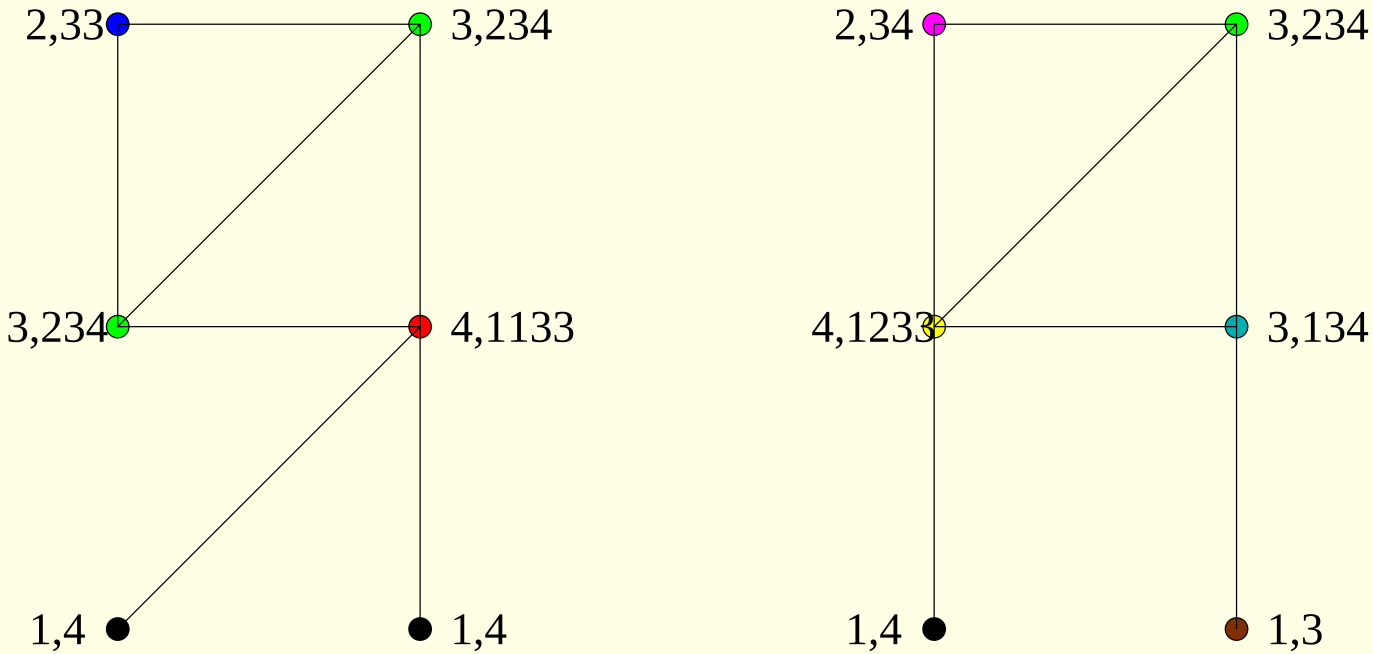
Refine coloring: for each vertex
New Color = Old Color + Old Colors of all neighbors

1-dim WL = color refinement procedure



Simplify color names

1-dim WL = color refinement procedure



Refine coloring again.

The multisets of colors differ,
hence the graphs are non-isomorphic.

1-dim WL $=$ color refinement procedure

k-dim WL $=$ the same idea, but now we color V^k instead of V. The initial coloring of $(v_1, \ldots, v_k)$ is the isomorphism type of the subgraph induced on $v_1, \ldots, v_k$.

Logical approach

Definition. Let $\sim$ denote the adjacency relation. A first order formula Φ over $\{\sim, =\}$ distinguishes G and H if it is true on exactly one of the two graphs.

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Cai, Fürer, and Immerman (FOCS'89): If G and H are distinguishable by a Φ with k variables and quantifier depth r, then the $(k-1)$-dim WL distinguishes G and H after r color refinement steps.

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Grohe and V. (ICALP'06): If any non-isomorphic n-vertex G and H in a class $\mathcal{C}$ are distinguishable with $O(1)$ variables and quantifier depth $O(\log n)$, then Graph Isomorphism for $\mathcal{C}$ is solvable in AC^1 by a variant of the $(k-1)$-dim WL algorithm.

Our result

Theorem. Every 3-connected planar graph on n vertices is definable (=distinguishable from any other graph) with 15 variabels and quantifier depth $11 \log_2 n + 45$.

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Earlier result (Grohe, LICS'98) Every planar graph is definable with $O(1)$ variables in the first order logic with counting quantifiers.

Proof ingredient I

Immerman-Poizat: G and H are distinguishable with k variables and quantifier depth r iff Spoiler wins the Ehrenfeucht game with k pebbles in r moves.

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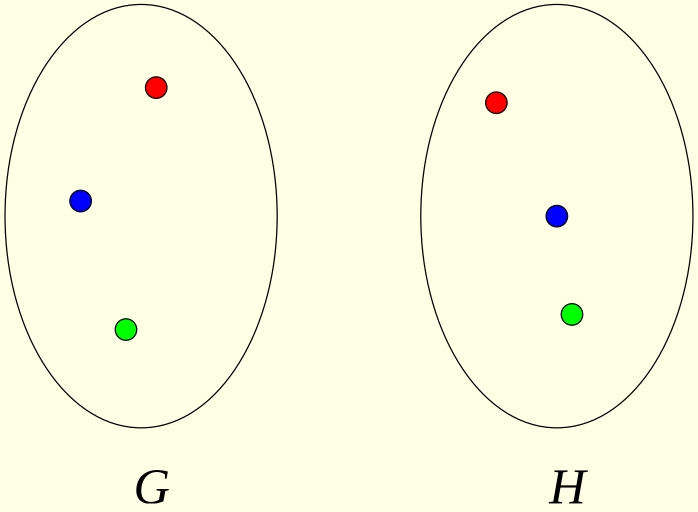
Rules of the Game



Players: Spoiler and Duplicator
Resources: k pebbles, each in duplicate

A round:
Spoiler puts a pebble on a vertex in G or H
Duplicator puts the other copy on a vertex in the other graph

Duplicator's objective: after each round the pebbling should determine a partial isomorphism between G and H

Proof ingredient II

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Grohe and V. (ICALP'06):

If $R_G \ncong R_H$, then Spoiler wins the Ehrenfeucht game on R_G and R_H with 5 pebbles in $3\log_2 n + 8$ moves.

`Pictures borrowed from cs.smu.ca/~dawson`

Proof ingredient III

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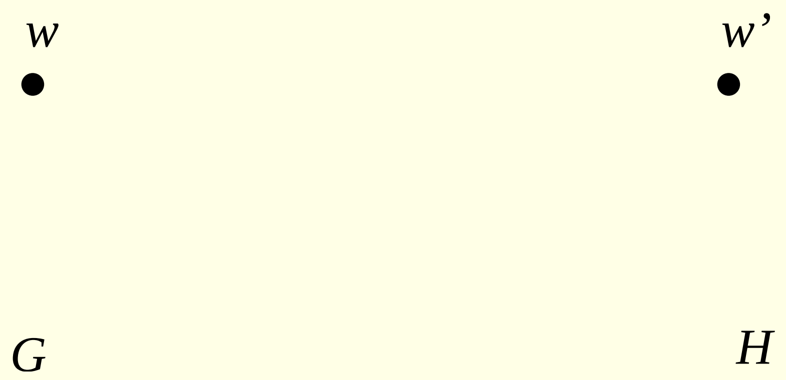
Let pebbles x, y, u, v, and w be placed in G and their copies x', y', u', v', and w' be placed in H.



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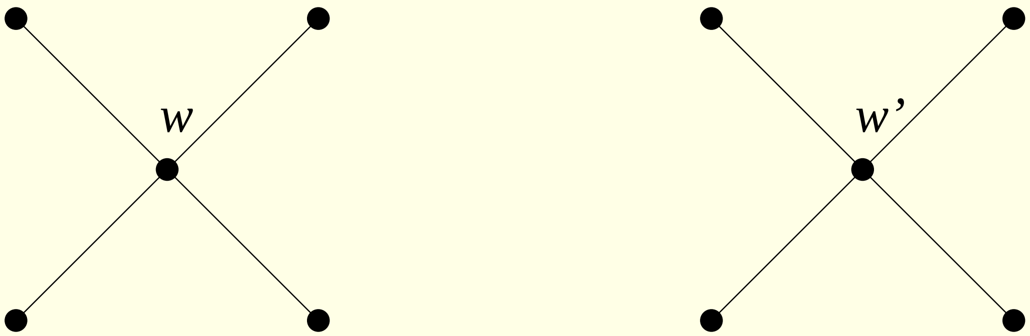

G H

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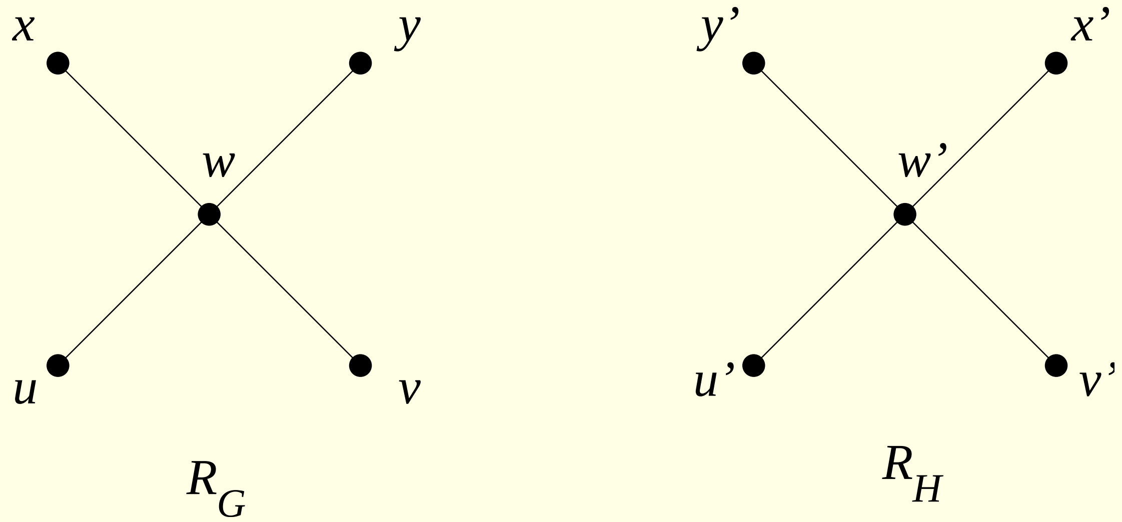

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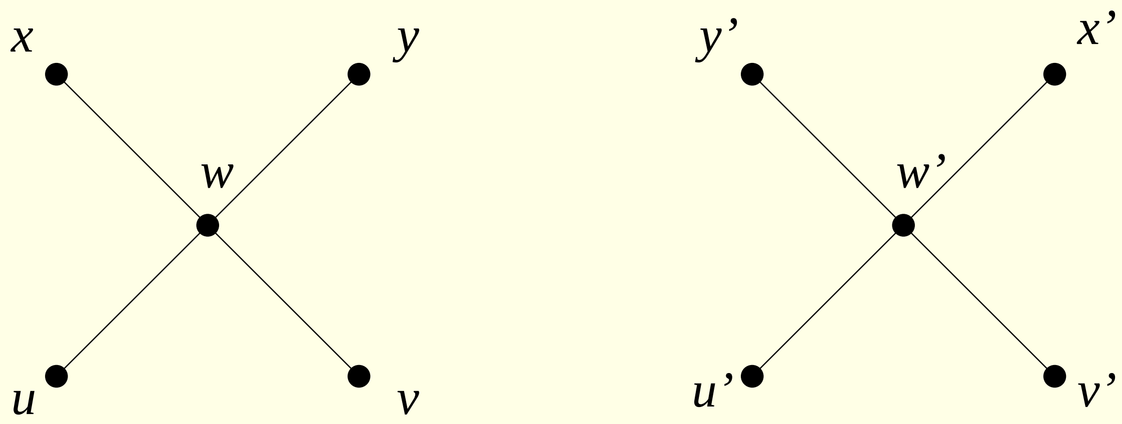

a collocated configuration a twisted configuration

The Simulation Lemma.
Starting from such a position, Spoiler wins the Ehrenfeucht game on G and H with 15 pebbles in $6 \log_2 n + 26$ moves.

Simulation Lemma: Proof in the simplest case

Let d denote the distance in graphs. Duplicator is forced to respect d: If, say, $d(x, y) < d(x', y')$, then Spoiler wins with 3 pebbles in $\lceil \log_2 d(x, y) \rceil$ using the halving strategy.

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Given $a, b \in V(G)$, let $d_0(a, b)$ be equal to the minimum length of a path from a to b not going through x, y, u, v, w (similarly for H). Duplicator should respect d_0 as well.

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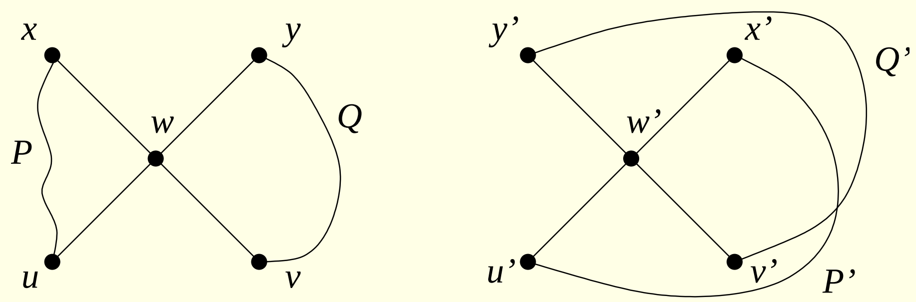

Assumption: Every of these four distances is attained by a single path, P, P', Q, and Q' respectively. P and Q do not intersect.

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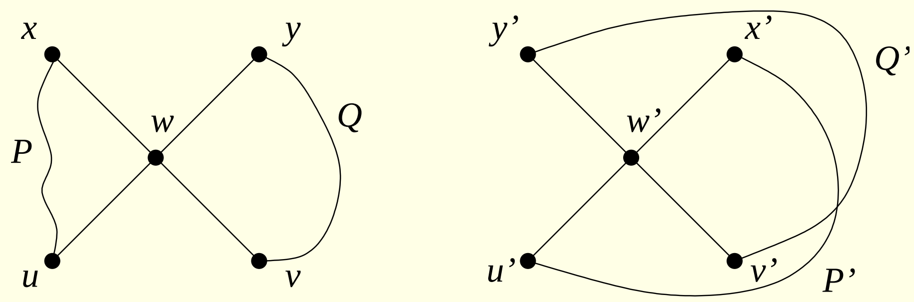

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Spoiler forces violating d_0 by pebbling a common point of P' and Q'.

Thank you!