Bounded-Variable Fragments of Hybrid Logics

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joint work with Thomas Schwentick

Propositional Modal Logic

Syntax Given a set of proposition symbols $\{p, q, \ldots\}$ and a modality $\mathbf{F}$, the basic Modal Logic $\mathcal{ML}$ is defined as follows

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid \mathbf{F}\varphi$$

- Abbreviations $\vee$, $\rightarrow$, $\leftrightarrow$ as usual; $\mathbf{G}\varphi = \neg\mathbf{F}\neg\varphi$

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Semantics via Kripke models $\mathcal{M} = (S, R, V)$

- S is a non-empty set of states
- R is a binary relation on S, the accessibility relation
- V is a valuation assigning subsets of S to proposition symbols

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What $\mathcal{ML}$ and $\mathcal{TL}$ cannot do: Talk about individual states

Hybrid Logic

Nominals

- A second sort of atomic formula
- Each nominal is true at exactly one state in any model
- I.e., a nominal names a state
- Nominals correspond to constants in first-order logic

$\mathcal{HL} = \mathcal{ML}$ plus nominals, $\mathcal{HTL} = \mathcal{TL}$ plus nominals

The root of a tree

- Name the root by a nominal *root*

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The @-Operator

- Jumps to a named state

The root of a tree

- Name the root by a nominal *root*
- Talk about the root: $@_{root}\varphi$

Hybrid Logic

The $\downarrow$-Operator

- Binds state variables to states
- $\downarrow x.\varphi \equiv$ name the current state x and evaluate φ
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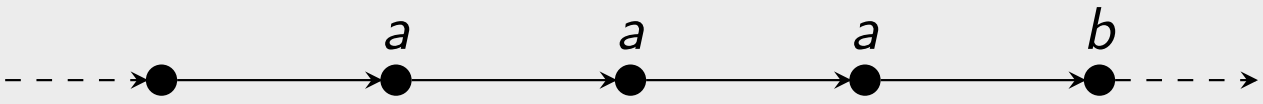
Expressing UNTIL with $\downarrow$ and @

- $a\mathbf{U}b$



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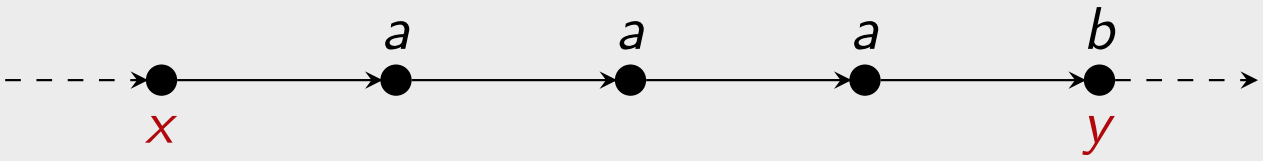

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Complexity of Hybrid Satisfiability Problems

models	$\mathcal{HL}(\downarrow)$	$\mathcal{HL}(\downarrow, @)$	$\mathcal{HTL}(\downarrow)$	$\mathcal{HTL}(\downarrow, @)$
arbitrary	undec.	undec.	undec.	undec.
linear / $(\mathbb{N}, <)$	**NP**	non-elem.	non-elem.	non-elem.

- Areces, Blackburn, Marx 1999
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Are there decidable fragments / fragments of lower complexity?

- ten Cate, Franceschet 2005
  - Disallowed patterns, e.g. $\mathbf{G} \downarrow \mathbf{G}$
  - Models of bounded width
- Our approach
  - Bounded-variable fragments

Bounded-Variable Fragments

- Only k different variable symbols for a fixed k
- $\mathcal{HL}(\downarrow^k)$, $\mathcal{HL}(\downarrow^k, @)$, $\mathcal{HTL}(\downarrow^k)$, $\mathcal{HTL}(\downarrow^k, @)$

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$\mathcal{HL}(\downarrow^2, @)$ and $\mathcal{HTL}(\downarrow^1)$ are expressively complete over the natural numbers.

Proof: $\mathcal{LTL}$ is expressively complete [Gabbay et al. 1980, 1989]

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$\mathcal{HL}(\downarrow^1, @)$ is expressively complete over the natural numbers with zero $(\mathbb{N}, <, 0)$.

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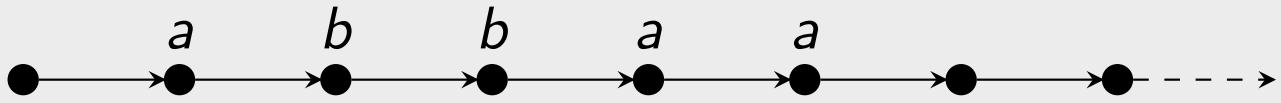
Proof: Reduction from the non-emptiness problem for star-free expressions (union, concatenation, and negation).

The string *abbaa*



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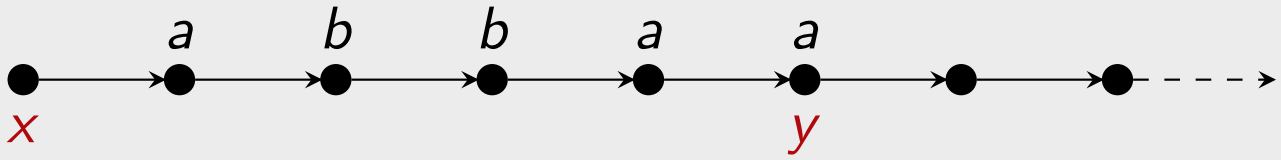

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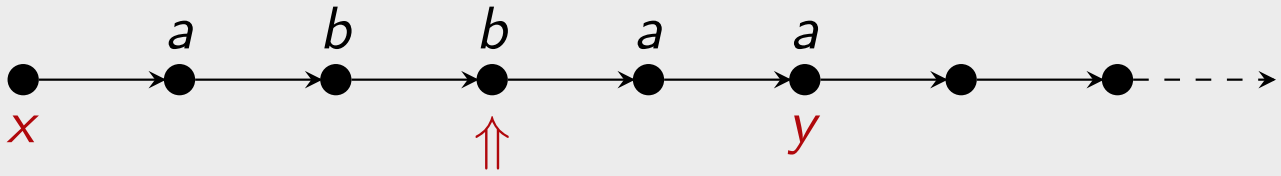

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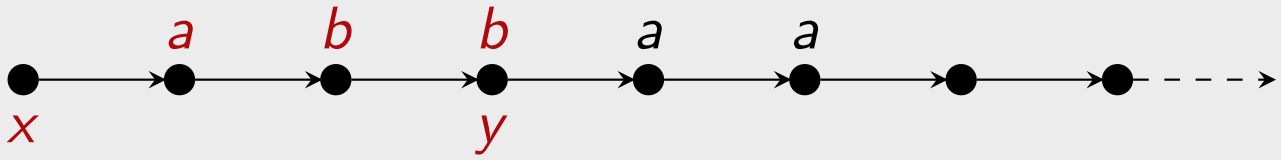

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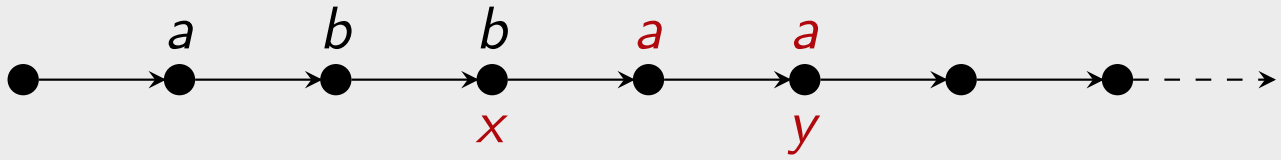

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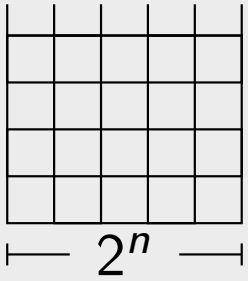

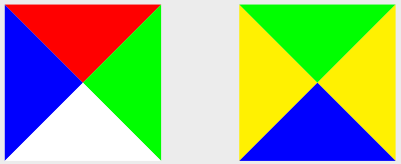

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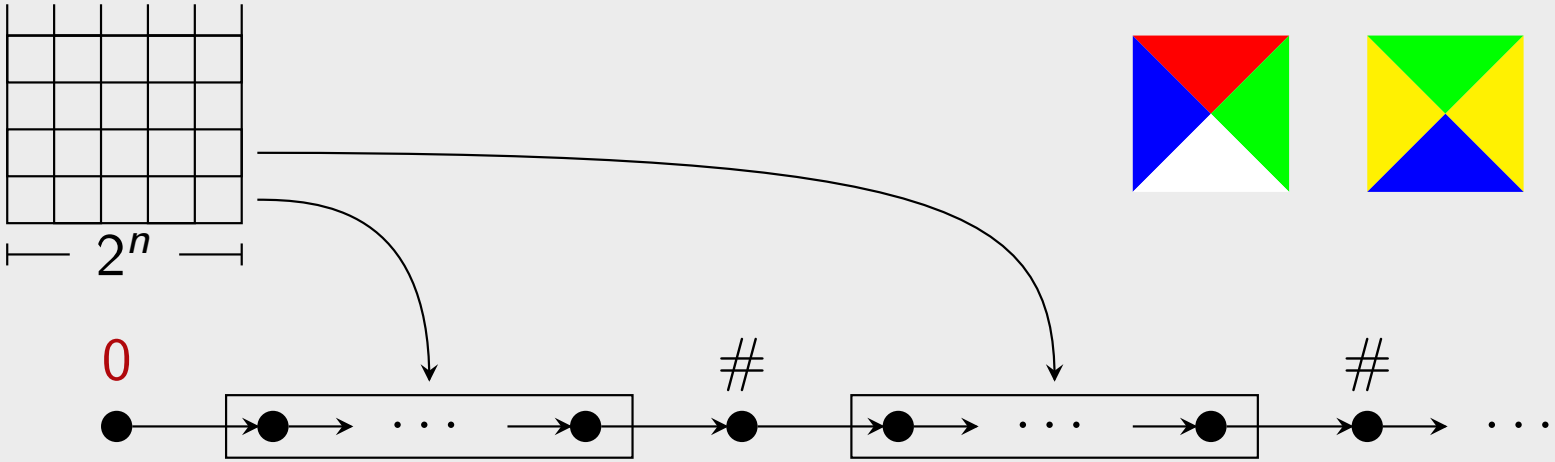

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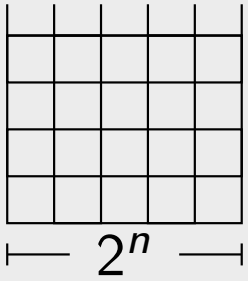
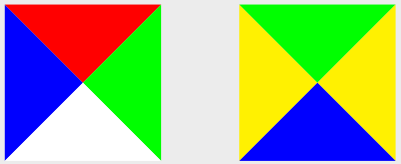
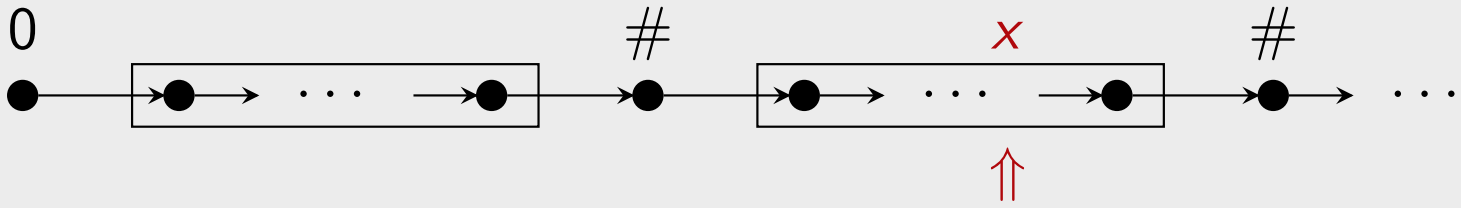


$\downarrow x.$





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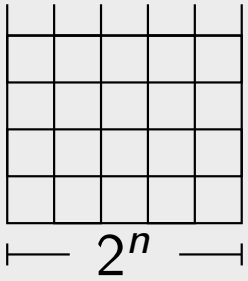

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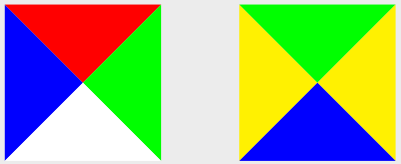

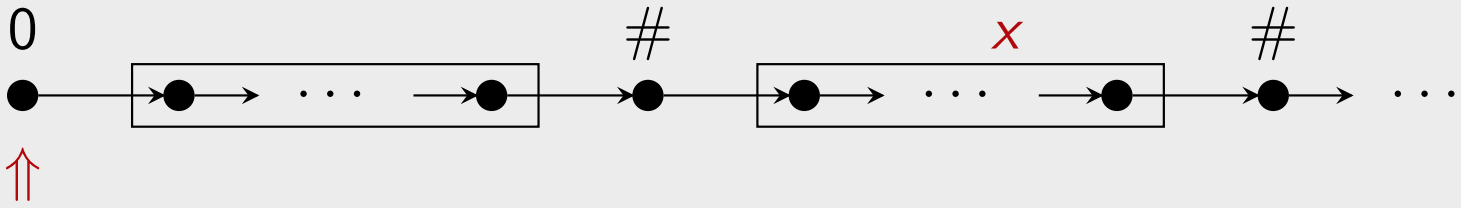

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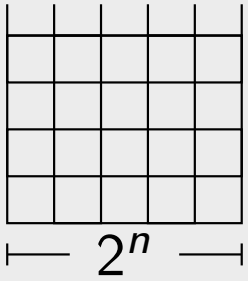

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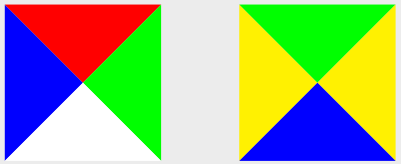

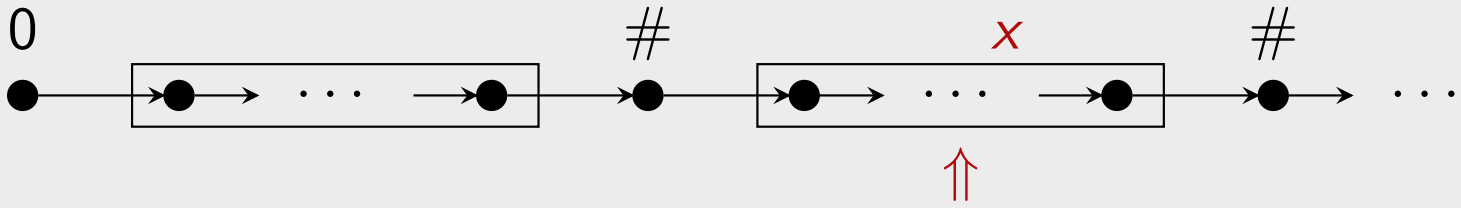

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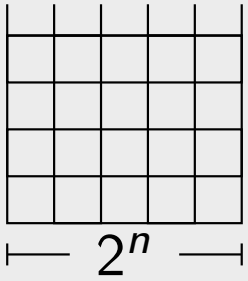

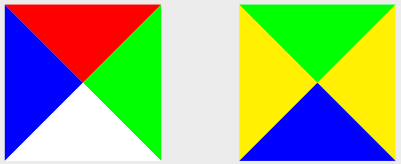

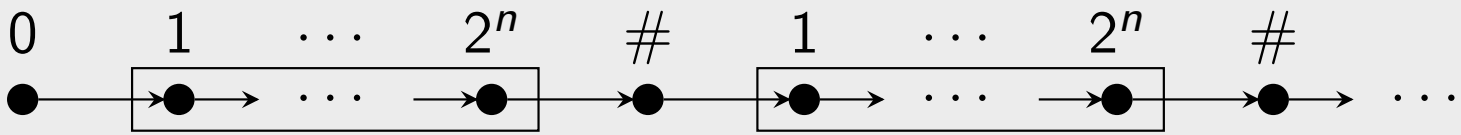

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Upper Bound: **Automata-theoretic approach**

- alternating Büchi automata and $\mathcal{LTL}$ [Vardi et al. since 1994]
- temporal logic $\rightarrow$ two-wayness
- $\downarrow^1 \rightarrow$ one pebble

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Upper Bound:

- Existence of **memoryless runs**
  - accepting run $\equiv$ winning strategy in a two-player game on the configuration graph
  - existence of memoryless winning strategies [Emerson, Jutla 91]
  - Thanks to an anonymous referee for this nice proof
- Simulation by **non-deterministic one-way Büchi automata** of double exponential size
  - checks existence of a memoryless accepting run
  - uses behavioral functions
  - in contrast to the finite case: local consistency does not suffice

Conclusion

- Hybrid $\downarrow$-languages are undecidable / have non-elementary complexity in general.

$\rightarrow$ **Bounded-variable fragments over the natural numbers**

- The one-variable fragment
  - Has full expressive power
  - Model checking can be done in polynomial time
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What could be next?

- Logic: extend results to trees

$\rightarrow$ same for automata (Conjecture: **2EXPTIME**)

- Automata: extend results to k-pebble Büchi automata