Adding Monotonic Counters to Automata and Transition Graphs

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Lehrstuhl für Informatik VII
Motivations: Parikh Automata (Klaedtke & Rueß, ICALP 2003)

Idea: how to recognize a language with arithmetical properties, such as
$L_{abc} := \{a^k b^k c^k \mid k \geq 1\}$?
Idea: how to recognize a language with *arithmetical properties*, such as
\[ L_{abc} := \{a^k b^k c^k \mid k \geq 1 \}? \]

- Use a finite automaton recognizing \(a^+ b^+ c^+\).
- Assign a *vector* to each input symbol.
- Put a Presburger *constraint* on the summed vector \((x, y, z)\):
  \[ x = y = z. \]

![Diagram of a finite automaton recognizing \(a^k b^k c^k\)]
1. Parikh automata
   - Semi-linear sets and Parikh’s theorem
   - Parikh automata and the Chomsky hierarchy

2. Monotonic-counter extensions of (infinite) graphs
   - Some classes of infinite graphs
   - Monotonic-counter extensions

3. Reachability problem
Part 1

Parikh Automata
A \subseteq \mathbb{N}^n \text{ linear: } A = \{ \bar{x}_0 + k_1 \bar{x}_1 + \ldots + k_m \bar{x}_m \mid k_1, \ldots, k_m \in \mathbb{N} \}

for some \( \bar{x}_0, \bar{x}_1, \ldots, \bar{x}_m \in \mathbb{N}^n \)

\textbf{Semi-linear set:} finite union of linear sets.

Example: \( B := \{ (x_1, x_2, x_3) \in \mathbb{N}^3 \mid x_1 < x_2 < x_3 \} \) is linear:

\[
\{ (0, 1, 2) + k_1(0, 0, 1) + k_2(0, 1, 1) + k_3(1, 1, 1) \mid k_1, k_2, k_3 \in \mathbb{N} \}.
\]
Semi-Linear Sets and Parikh’s Theorem

A ⊆ N^n linear: \( A = \{ \bar{x}_0 + k_1 \bar{x}_1 + \ldots + k_m \bar{x}_m \mid k_1, \ldots, k_m \in \mathbb{N} \} \)
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Properties of semi-linear sets:

- effective closure under Boolean operations [Ginsburg & Spanier]
- equivalence to Presburger-definable sets [Ginsburg & Spanier]
Semi-Linear Sets and Parikh’s Theorem

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**Parikh’s theorem**: The Parikh image of any context-free language is effectively semi-linear.
**Parikh Automata**

*Parikh finite automaton* (Parikh-FA) \((\mathfrak{A}, C)\) of dimension \(n \geq 1\) over \(\Sigma\):

- finite automaton \(\mathfrak{A}\) over \(\Sigma \times D\) \((D \subseteq \mathbb{N}^n\) finite, nonempty) 

- semi-linear set \(C \subseteq \mathbb{N}^n\)
Parikh Automata

Parikh finite automaton (Parikh-FA) \((\mathfrak{A}, C)\) of dimension \(n \geq 1\) over \(\Sigma\):

- finite automaton \(\mathfrak{A}\) over \(\Sigma \times D\) \((D \subseteq \mathbb{N}^n\) finite, nonempty\)
- semi-linear set \(C \subseteq \mathbb{N}^n\)

Word \(u := a_1 \cdots a_m\) is accepted iff

- \(v := (a_1, \bar{d}_1) \cdots (a_m, \bar{d}_m) \in L(\mathfrak{A})\) exists, for some \(\bar{d}_1, \ldots, \bar{d}_m \in D\),
- and \(\Phi(v) := \bar{d}_1 + \cdots + \bar{d}_m \in C\).

extended Parikh mapping \(\Phi: (\Sigma \times D)^* \rightarrow \mathbb{N}^n\)
Emptiness Problem

Lemma (Klaedtke & Rueß):
If the *Parikh image* of $L \subseteq (\Sigma \times D)^*$ is effectively semi-linear, then also its *extended Parikh image* $\Phi(L)$.

Theorem (Klaedtke & Rueß):
The emptiness problem for Parikh-FA’s is decidable.

*Proof idea.*

\[ L(A, C) \neq \emptyset \quad \text{iff} \quad \Phi(L(A)) \cap C \neq \emptyset \]

- Both sets are semi-linear.
- Intersection of semi-linear sets is effectively semi-linear.

$\implies$ (Non-)Emptiness is decidable.
Automata of the Chomsky hierarchy as the automaton component $\mathcal{A}$:

- Turing machines
  - linear-bounded automata
  - pushdown automata
  - finite automata
Automata of the Chomsky hierarchy as the automaton component $\mathfrak{A}$:
Automata of the Chomsky hierarchy as the automaton component \( \mathfrak{A} \):

1. \( \{a^k b^k c^k | k \geq 1\} \)
2. \( \{ww^R | w \in \{a, b\}^*\} \)
3. semi-linearity of Parikh-PDA recognizable languages
Part 2

Monotonic-Counter Extensions of (Infinite) Graphs
**Σ-labeled graph**

\[ G := (V, (E_a)_{a \in \Sigma}) \]
Monotonic-Counter Graphs

\((\Sigma \times D)\)-labeled graph

\[ G := (V, (E_{(a, \bar{a})})_{(a, \bar{a}) \in \Sigma \times D}) \]

\((D \subseteq \mathbb{N}^n \text{ finite, nonempty})\)

\[ (a, (1, 0)) \quad (b, (0, 1)) \]
Monotonic-Counter Graphs

$(\Sigma \times D)$-labeled graph

$G := (V, (E(a, \bar{d}))_{(a, \bar{d}) \in \Sigma \times D})$

$(D \subseteq \mathbb{N}^n$ finite, nonempty$)$

Monotonic-counter extension of $G$:

$\Sigma$-labeled graph $\tilde{G} := (\tilde{V}, (\tilde{E}_a)_{a \in \Sigma})$ with

$\tilde{V} := V \times \mathbb{N}^n$ and $((\alpha, \bar{x}), (\beta, \bar{y})) \in \tilde{E}_a$ iff

$(\alpha, \beta) \in E(a, \bar{d})$ and

$\bar{y} = \bar{x} + \bar{d}$, for some $\bar{d} \in D$. 

(a, (1, 0)) (b, (0, 1))

(0, 0) \rightarrow (1, 0) \rightarrow (2, 0) \rightarrow \ldots

\begin{align*}
\downarrow b \\
\downarrow a \\
(0, 1) \rightarrow (1, 1) \rightarrow (2, 1) \rightarrow \ldots
\end{align*}

\begin{align*}
\downarrow b \\
\downarrow a \\
(0, 2) \rightarrow (1, 2) \rightarrow (2, 2) \rightarrow \ldots
\end{align*}

\begin{align*}
\downarrow b \\
\downarrow a \\
\downarrow b \\
\ldots
\end{align*}
Some Classes of Infinite Graphs with Finite Representations

Vertices: regular sets over alphabet $\Gamma$

Edges: automaton-definable relations over words, e.g.:

- **pushdown graphs** [Muller & Schupp]: transitions of $\varepsilon$-free pushdown automata

- **prefix-recognizable graphs** [Caucal]: generalized prefix rewriting rules

- **synchronized rational graphs or automatic graphs**
  [Frougny & Sakarovitch, Blumensath & Grädel]: synchronized rational relations

- **rational graphs** [Morvan]: rational relations
Hierarchy of Graph Classes

finite graphs
pushdown graphs
prefix-recognizable graphs
synchronized rational graphs
rational graphs
Hierarchy of Graph Classes

- finite graphs
  - pushdown graphs
    - prefix-recognizable graphs
      - synchronized rational graphs
        - rational graphs
          - prex-recognizable graphs
Hierarchy of Graph Classes

1. infinite two-dimensional grid
2. decidability of the reachability problem
3. graphs with vertices of unbounded degree
4. graphs with repetition-free cycles of unbounded length
Synchronized Rational Graphs

Vertices: regular set $V$ over alphabet $\Gamma$

Edges: synchronized rational relations, i.e. edge relation $E_a$ is recognized by a \textit{finite-state} automaton working on pairs $(X_1 \cdots X_m, Y_1 \cdots Y_n) \in \Gamma^* \times \Gamma^*$ with two \textit{one-way} input tapes and \textit{simultaneously} moving input heads.

\begin{center}
\begin{tabular}{ccccccc}
$X_1$ & $X_2$ & $X_3$ & $\cdots$ & $X_m$ & $\diamond$ & $\cdots$ & $\diamond$
\end{tabular}
\end{center}

\begin{center}
\begin{tabular}{ccccccc}
$Y_1$ & $Y_2$ & $Y_3$ & $\cdots$ & $Y_m$ & $Y_{m+1}$ & $\cdots$ & $Y_n$
\end{tabular}
\end{center}

finite memory

new symbol
Synchronized Rational Graphs: Example

\[
\begin{array}{c}
\varepsilon \\
\downarrow b \\
Y \\
\downarrow c \\
YY \\
\downarrow c \\
YYY \\
\end{array}
\quad
\begin{array}{c}
a \\
\downarrow a \\
X \\
\downarrow b \\
XX \\
\downarrow a \\
XXX \\
\end{array}
\quad
\begin{array}{c}
b \\
\downarrow b \\
XY \\
\downarrow b \\
XYY \\
\downarrow b \\
XYYY \\
\end{array}
\quad
\begin{array}{c}
c \\
\downarrow c \\
Y \\
\downarrow c \\
Y Y Y Y \\
\downarrow c \\
X Y Y Y Y \\
\end{array}
\quad
\begin{array}{c}
\ldots \\
\ldots \\
\ldots \\
\ldots \\
\ldots \\
\end{array}
\]
Monotonic-Counter Extensions of Synchronized Rational Graphs

$(\Sigma \times D)$-labeled synchronized rational graph

$G := (V, (E_{(a,\bar{a})})_{(a,\bar{a}) \in \Sigma \times D})$

$(D \subseteq \mathbb{N}^n$ finite, nonempty)
Monotonic-Counter Extensions of Synchronized Rational Graphs

\((\Sigma \times D)\)-labeled synchronized rational graph

\[ G := (V, (E_{(a,\bar{d})})_{(a,\bar{d}) \in \Sigma \times D}) \]

\((D \subseteq \mathbb{N}^n \text{ finite, nonempty})\)

\(\Sigma\)-labeled SRMC graph

\[ \tilde{G} := (\tilde{V}, (\tilde{E}_a)_{a \in \Sigma}) \text{ with } \tilde{V} := V \times \mathbb{N}^n \]

and \(( (\alpha, \bar{x}), (\beta, \bar{y}) ) \in \tilde{E}_a \) iff

\[ (\alpha, \beta) \in E_{(a,\bar{d})} \text{ and } \bar{y} = \bar{x} + \bar{d}, \text{ for some } \bar{d} \in D. \]
Monotonic-Counter Extensions of Synchronized Rational Graphs

\[(\Sigma \times D)\text{-labeled synchronized rational graph}\]

\[
G := (V, (E_{(a,\bar{d})})_{(a,\bar{d}) \in \Sigma \times D})
\]

\[
(D \subseteq \mathbb{N}^n \text{ finite, nonempty})
\]

\[\Sigma\text{-labeled SRMC graph}\]

\[
\tilde{G} := (\tilde{V}, (\tilde{E}_a)_{a \in \Sigma}) \text{ with } \tilde{V} := V \times \mathbb{N}^n
\]

and \(((\alpha, \bar{x}), (\beta, \bar{y})) \in \tilde{E}_a \text{ iff}\)

\[
(\alpha, \beta) \in E_{(a,\bar{d})} \text{ and } \\
\bar{y} = \bar{x} + \bar{d}, \text{ for some } \bar{d} \in D.
\]

**Proposition:** \(\tilde{G}\) is synchronized rational.

**Proof sketch.** Encode vertex \((\alpha, (x_1, \ldots, x_n))\) of \(\tilde{G}\) by means of word

\[
\underbrace{\#_1 \cdots \#_1}_{x_1} \cdots \underbrace{\#_n \cdots \#_n}_{x_n} \alpha
\]

Define automaton for \(\widetilde{E}_a\) working on pairs \((\#_{x_1}^{x_1} \cdots \#_{x_n}^{x_n} \alpha, \#_{y_1}^{y_1} \cdots \#_{y_n}^{y_n} \beta)\):

1. Guess a vector \(\bar{d} \in D\) and check whether \(\bar{x} + \bar{d} = \bar{y}\).

2. Simulate the automaton for \(E_{(a,\bar{d})}\) on \((\alpha, \beta)\).

*Bounded delay* sufficient since \(D\) and \(\Gamma\) are finite. \(\square\)
Part 3

Reachability Problem
**Logical Decision Problems over Transition Graphs**

- **Reachability:**
  
  Given a graph $G$ and two vertices $\alpha$ and $\beta$ in $G$, is $\beta$ reachable from $\alpha$?

- **First-order (FO) theory:**
  
  Given a graph $G$ and a first-order sentence $\varphi$, does $\varphi$ hold in $G$?

- **Monadic second-order (MSO) theory:**
  
  Given a graph $G$ and a monadic second-order sentence $\varphi$, does $\varphi$ hold in $G$?
Logical Decision Problems over Transition Graphs

- FO theory
  - undecidable
  - decidable

- rational graphs

- synchronized rational graphs
  - PR-MC
  - PD-MC
  - F-MC

- prefix-recognizable graphs

- pushdown graphs

- finite graphs
Logical Decision Problems over Transition Graphs

- **FO theory**
  - rational graphs
    - undecidable
    - decidable
  - synchronized rational graphs
    - undecidable
    - decidable
- **MSO theory**
  - prefix-recognizable graphs
    - undecidable
    - decidable
  - pushdown graphs
    - undecidable
    - decidable
  - finite graphs
    - undecidable
    - decidable
Logical Decision Problems over Transition Graphs

- **rational graphs**
  - **FO theory**
    - undecidable
    - decidable
  - **reachability**
    - undecidable
    - decidable
  - **MSO theory**
    - undecidable
    - decidable
- **synchronized rational graphs**
  - **PR-MC**
  - **prefix-recognizable graphs**
    - **PD-MC**
    - **F-MC**
- **pushdown graphs**
- **finite graphs**
Prefix-Recognizable Graphs

\[ \Sigma := \{a, b\} \quad \Gamma := \{Z\} \]

\[ \text{rule } U \xrightarrow{a} V \text{ with } U, V \subseteq \Gamma^* \text{ regular} \]

Prefix-rewriting system \( R := \{ \varepsilon \xrightarrow{a} Z , \varepsilon \xrightarrow{b} Z^+ \} \)

Prefix-recognizable graph \( G = (V, E_a, E_b) \) defined by \( R \):

- \( V = \Gamma^* \),
- \( E_a = \{(Z^i, Z^{i+1}) \mid i \in \mathbb{N}\} \), and
- \( E_b = \{(Z^i, Z^j) \mid i, j \in \mathbb{N} \text{ and } i < j\} \).
A \((\Sigma \times D)\)-labeled prefix-recognizable graph

\[ G := (V, (E_{(a,\bar{d})})_{(a,\bar{d}) \in \Sigma \times D}) \]

\((D \subseteq \mathbb{N}^n \text{ finite, nonempty})\)
(\Sigma \times D)\text{-labeled prefix-recognizable graph} \quad \tilde{G} := (\tilde{V}, (\tilde{E}_a)_{a \in \Sigma}) \text{ with } \tilde{V} := V \times \mathbb{N}^n \\
\quad G := (V, (E_{(a,\bar{d})})_{(a,\bar{d}) \in \Sigma \times D}) \\
(D \subseteq \mathbb{N}^n \text{ finite, nonempty}) \\
\quad \tilde{V} := V \times \mathbb{N}^n \\
\quad \tilde{E}_a := \{((\alpha, \bar{x}), (\beta, \bar{y})) \in \tilde{E}_a \text{ iff } (\alpha, \beta) \in E_{(a,\bar{d})} \text{ and } \bar{y} = \bar{x} + \bar{d}, \text{ for some } \bar{d} \in D\}.
Monotonic-Counter Extensions of Prefix-Recognizable Graphs

\[(\Sigma \times D)\text{-labeled prefix-recognizable graph} \]
\[G := (V, (E_{(a,\bar{d})})_{(a,\bar{d}) \in \Sigma \times D}) \quad \sim \quad \Sigma\text{-labeled PRMC graph} \]
\[\tilde{G} := (\tilde{V}, (\tilde{E}_a)_{a \in \Sigma}) \text{ with } \tilde{V} := V \times \mathbb{N}^n\]
\[(D \subseteq \mathbb{N}^n \text{ finite, nonempty}) \quad \text{and } ((\alpha, \bar{x}), (\beta, \bar{y})) \in \tilde{E}_a \text{ iff}\]
\[(\alpha, \beta) \in E_{(a,\bar{d})} \text{ and } \bar{y} = \bar{x} + \bar{d}, \text{ for some } \bar{d} \in D.\]

Reachability problem for \(\tilde{G}\):

Given: regular sets \(U, U' \subseteq V\) of vertices in \(G\) and semi-linear sets \(C, C' \subseteq \mathbb{N}^n\)

Question: are there vertices \((\alpha, \bar{x}) \in U \times C\) and \((\beta, \bar{y}) \in U' \times C'\) in \(\tilde{G}\) such that \((\beta, \bar{y})\) is reachable from \((\alpha, \bar{x})\)?
Proposition: The reachability problem for the monotonic-counter extension of any prefix-recognizable graph is decidable.

Proof sketch. Let \( L \subseteq (\Sigma \times D)^* \) be the traces of \( G \) with \( U \) and \( U' \) as the set of initial and final vertices, respectively.

Show

\[ U' \times C' \text{ is reachable from } U \times C \iff (C + \Phi(L)) \cap C' \neq \emptyset \]

Right-hand side:
- \( L \) is context-free and effectively constructible (Cauca 2003).
- \( \Phi(L) \) is effectively semi-linear.
- Effective closure of semi-linear sets under \( + \) and \( \cap \).
- \( \Rightarrow \) (Non-)Emptiness is decidable
Parikh automata correspond to adding monotonic counters with an additional semi-linearity test.

No increase in language recognition power for linear-bounded automata, in contrast to finite and pushdown automata.

Application to transition graphs: no increase for synchronized rational graphs, but for pushdown and prefix-recognizable graphs.

For prefix-recognizable graphs, reachability remains decidable.
Further Prospects

- Using more general frameworks than monotonic counters, e.g. reversal-bounded counters.
- Allowing intermediate tests during computations.
- Comparing Parikh automata to reversal-bounded counter automata.
- Using more general arithmetical conditions than semi-linear sets.