

A Computational Analysis of Minimal Unidirectional Covering Sets

Dorothea Baumeister, Felix Brandt,
Felix Fischer, Jan Hoffmann, and Jörg Rothe

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Outline

- 1 Solution Concepts
- 2 Unidirectional Covering
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Solution Concepts

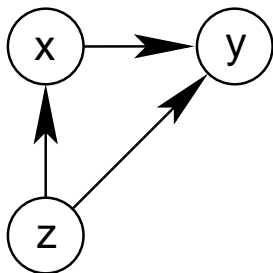
Binary dominance relations

Identify the “most desirable” elements in a pairwise majority relation:

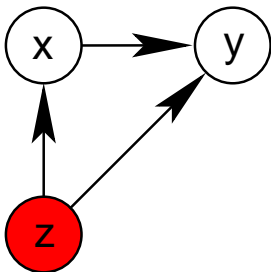
- game theory
- social choice theory
- argumentation theory
- sports tournaments
- ...

Natural concept: Choose the maximal element.

Example



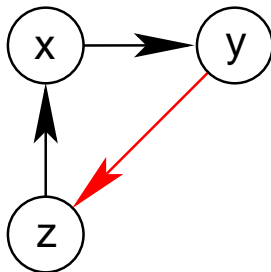
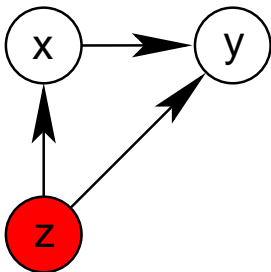
Example



Maximal element

z is the winner.

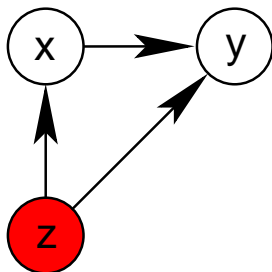
Example



Maximal element

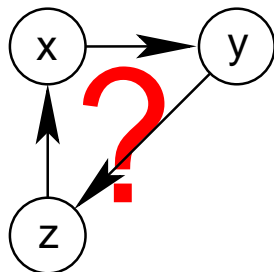
z is the winner.

Example



Maximal element

z is the winner.



Maximal element

There is no winner!

Condorcet's Paradox renders maximality useless
⇒ solution concepts

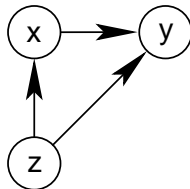
Solution Concept: Minimal Unidirectional Covering Sets

Unidirectional Covering

Let A be a finite set of alternatives, $B \subseteq A$, $\succ \subseteq A \times A$ a dominance relation, and let $x, y \in B$.

- x upward covers y ($x C_u y$) if $x \succ y$ and for all $z \in B$, $z \succ x$ implies $z \succ y$.

$x C_u y$, $z C_u x$, and $z C_u y$



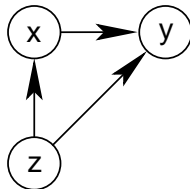
Solution Concept: Minimal Unidirectional Covering Sets

Unidirectional Covering

Let A be a finite set of alternatives, $B \subseteq A$, $\succ \subseteq A \times A$ a dominance relation, and let $x, y \in B$.

- **x upward covers y** ($x C_u y$) if $x \succ y$ and for all $z \in B$, $z \succ x$ implies $z \succ y$.
- **x downward covers y** ($x C_d y$) if $x \succ y$ and for all $z \in B$, $y \succ z$ implies $x \succ z$.

$x C_u y$, $z C_u x$, and $z C_u y$
 $z C_d x$, $z C_d y$, and $x C_d y$



Solution Concept: Minimal Unidirectional Covering Sets

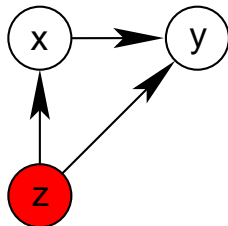
Uncovered Set

Let A be a finite set of alternatives, $B \subseteq A$, $\succ \subseteq A \times A$ a dominance relation, and let C be a covering relation on A . The **uncovered set of B** with respect to C is:

$$UC_C(B) = \{x \in B \mid yCx \text{ for no } y \in B\}.$$

$$UC_u(\{x, y, z\}) = \{z\}$$

$$UC_d(\{x, y, z\}) = \{z\}$$



Solution Concept: Minimal Unidirectional Covering Sets

Minimal Covering Set

Let A be a finite set of alternatives, $\succ \subseteq A \times A$ a dominance relation, and C a covering relation. $B \subseteq A$ is a covering set for A under C , if:

- $UC_C(B) = B$ (internal stability), and
- for all $x \in A - B$, $x \notin UC_C(B \cup \{x\})$ (external stability).

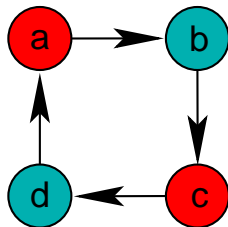
Such a B is **minimal** if no $B' \subset B$ is a covering set for A under C .

Minimal upward covering sets:

$$B_1 = \{a, c\} \text{ and } B_2 = \{b, d\}$$

Minimal downward covering set:

$$B_3 = \{a, b, c, d\}$$



Minimal Upward Covering Set Member

Definition

Name: Minimal Upward Covering Set Member (MC_U -Member).

Instance: A set A of alternatives, a dominance relation \succ on A , and a distinguished element $d \in A$.

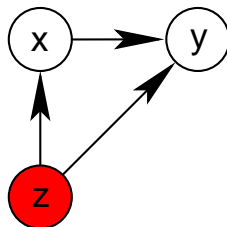
Question: Is d contained in some minimal upward covering set for A ?

$$A = \{x, y, z\}$$

$$\succ = \{(z, x), (z, y), (x, y)\}$$

$$(A, \succ, z) \in MC_U\text{-Member}$$

$$(A, \succ, x) \notin MC_U\text{-Member}$$



Unidirectional Covering Set Problems

- **MC_u-Size**: Given a set A of alternatives, a dominance relation \succ on A , and a positive integer k , does there exist some minimal upward covering set for A containing at most k alternatives?
- **MC_u-Member-All**: Given a set A of alternatives, a dominance relation \succ on A , and a distinguished element $d \in A$, is d contained in all minimal upward covering sets for A ?
- **MC_u-Unique**: Given a set A of alternatives and a dominance relation \succ on A , does there exist a unique minimal upward covering set for A ?
- **MC_u-Test**: Given a set A of alternatives, a dominance relation \succ on A , and a subset $M \subseteq A$, is M a minimal upward covering set for A ?
- **MC_u-Find**: Given a set A of alternatives and a dominance relation \succ on A , find a minimal upward covering set for A .

Minimality versus Minimum Size

Set-inclusion Minimality versus Minimum Cardinality

- **cardinality**: classical problems (maximum-size independent set, minimum-size dominating set, etc.)
- **set inclusion**: minimal upward covering set member.

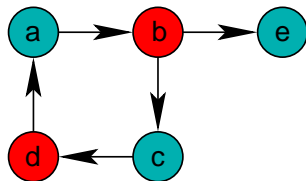
⇒ Standard techniques are not directly applicable.

Upward covering sets:

$$S = \{a, c, e\}$$

$$T = \{b, d\}$$

set inclusion minimal: S and T
cardinality minimal: only T



Lower Bound

Approach for proving Θ_2^P -hardness

NP-hardness

coNP-hardness

DP-hardness

Θ_2^P -hardness



NP-Hardness

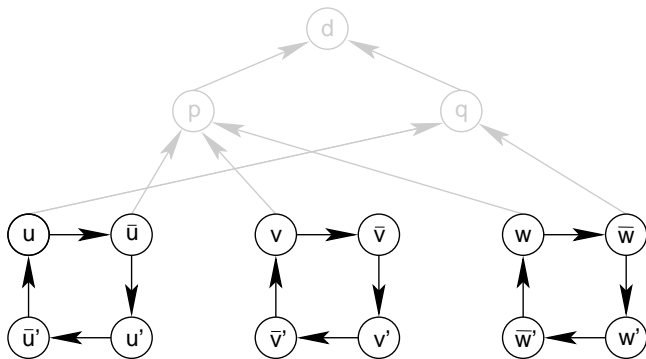
Reduction from SAT to MC_U -Member

There is a satisfying assignment for φ

\Leftrightarrow

there is a minimal upward covering set that contains d .

$$\varphi = (\bar{u} \vee v \vee w) \wedge (u \vee \bar{w})$$



Example: NP-Hardness

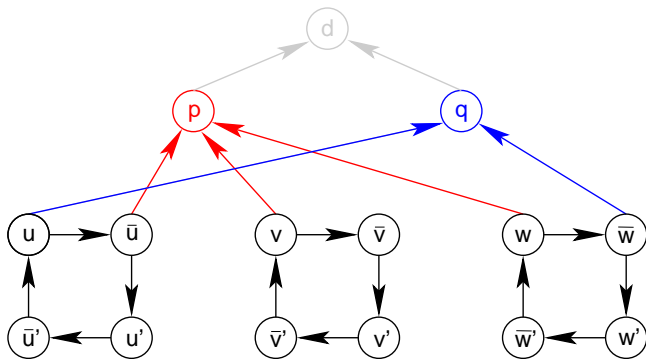
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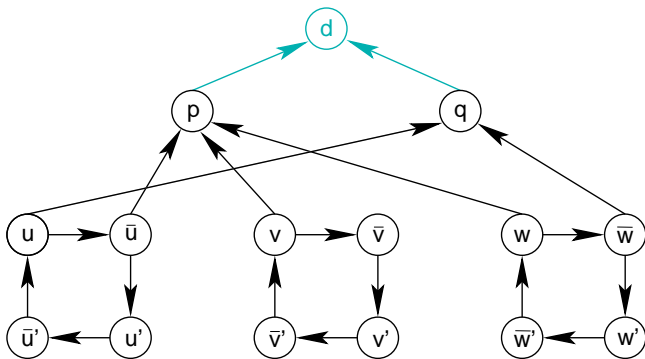
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Example: NP-Hardness

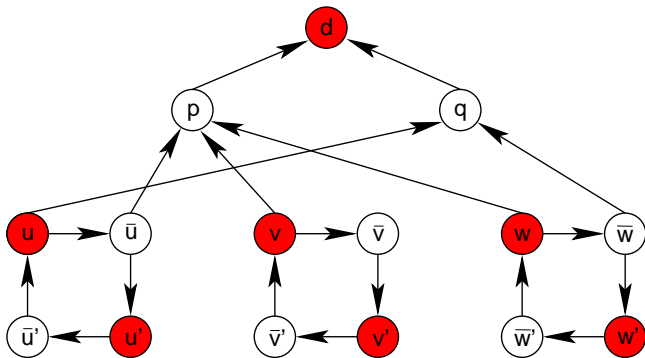
Reduction from SAT to MC_U -Member

There is a satisfying assignment for φ

\Leftrightarrow

there is a minimal upward covering set that contains d .

$\varphi = (\bar{u} \vee v \vee w) \wedge (u \vee \bar{w})$, satisfying assignment: $u = v = w = 1$



coNP-Hardness

The class coNP

Class of sets whose complements are in NP.

Reduction from SAT to the complement of MC_{\uparrow} -Member

There is a satisfying assignment for ψ

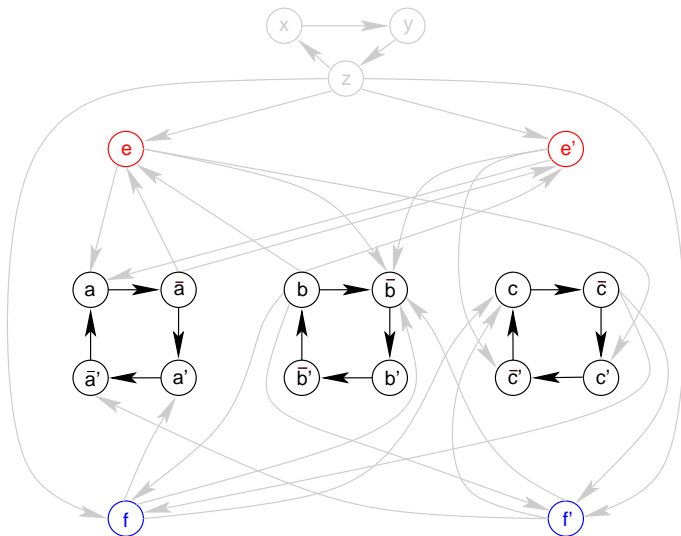
\Leftrightarrow

there is no minimal upward covering set that contains e .

Additionally: e is contained in all minimal upward covering sets if and only if there is no satisfying assignment for ψ .

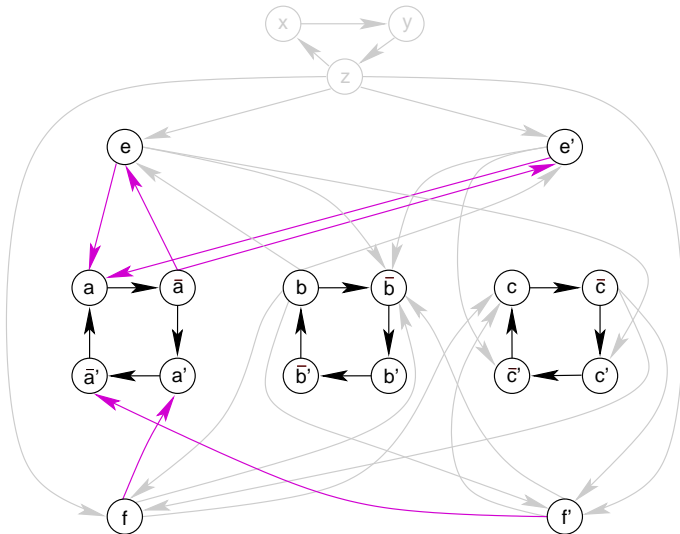
Example: coNP-Hardness

$$\psi = (\bar{a} \vee b) \wedge (b \vee \bar{c})$$



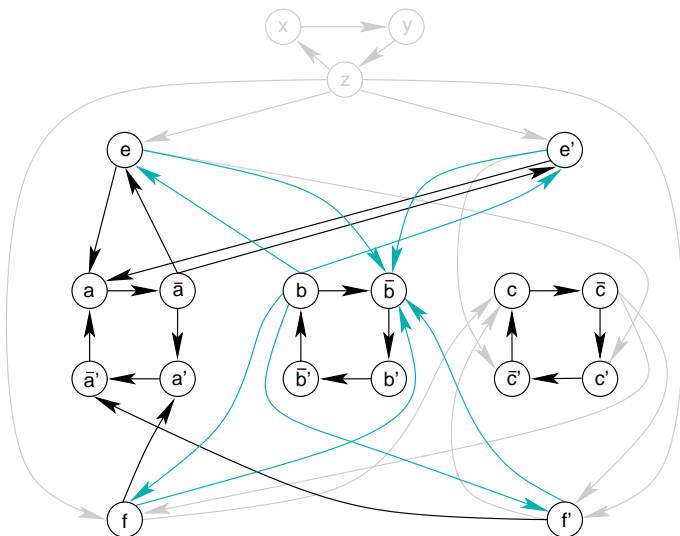
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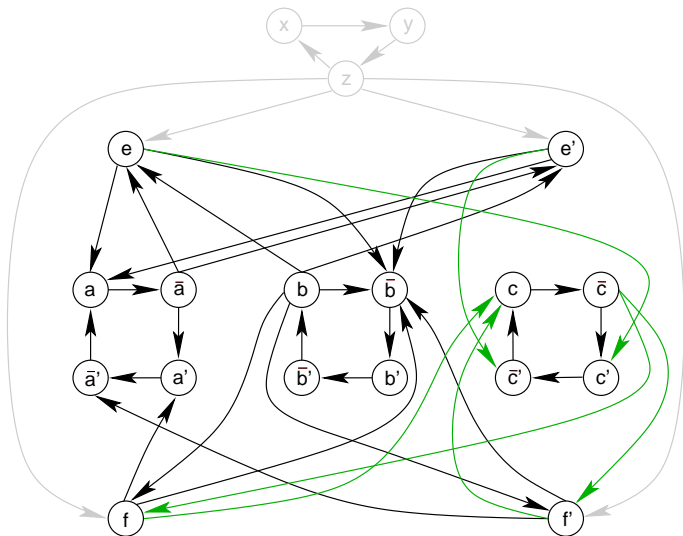
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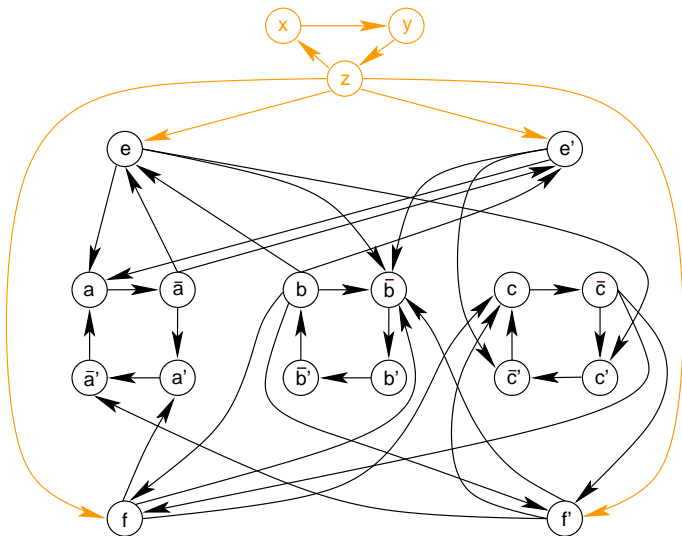
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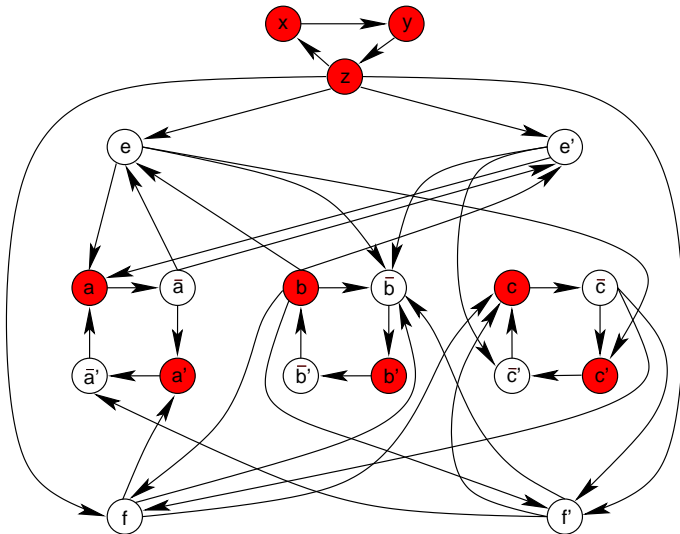
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Example: coNP-Hardness

$\psi = (\bar{a} \vee b) \wedge (b \vee \bar{c})$, satisfying assignment: $a = b = c = 1$



DP-Hardness

The class DP

The class of differences of two NP sets: $DP = \{A - B \mid A, B \in NP\}$.

$$NP \cup \text{coNP} \subseteq DP.$$

Wagner's Lemma for DP-Hardness

Let A be some NP-complete problem, let B be an arbitrary problem. If there exists a polynomial-time computable function f such that, for all strings x_1, x_2 satisfying that if $x_2 \in A$ then $x_1 \in A$, it holds:

$$(x_1 \in A \text{ and } x_2 \notin A) \Leftrightarrow f(x_1, x_2) \in B,$$

then B is DP-hard.

Construction

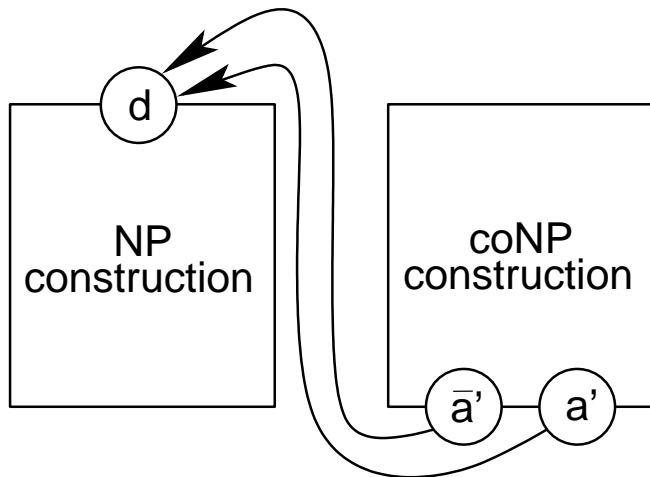
There is a satisfying assignment for φ , and none for ψ

$$\Leftrightarrow$$

there is a minimal upward covering set that contains d .

Proof Sketch: DP-Hardness

Combination of the previously presented NP and coNP reductions.



Θ_2^P -Hardness

The class Θ_2^P

Θ_2^P (also known as $P_{||}^{NP}$) is the class of problems solvable by a polynomial-time algorithm having parallel access to an NP oracle.

$$NP \cup \text{coNP} \subseteq DP \subseteq \Theta_2^P.$$

Wagner's Lemma for Θ_2^P -Hardness

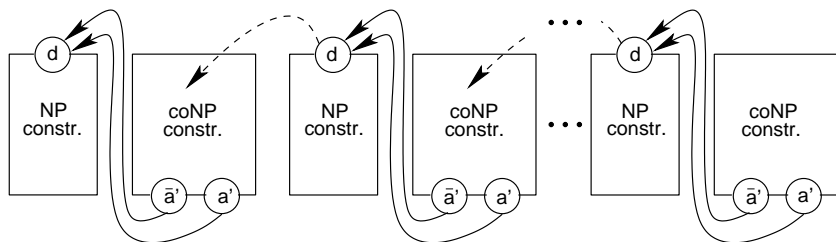
Let A be some NP-complete problem, and let B be an arbitrary problem. If there exists a polynomial-time computable function f such that, for all $m \geq 1$ and all strings x_1, x_2, \dots, x_{2m} satisfying that if $x_j \in A$ then $x_{j-1} \in A$, $1 < j \leq 2m$, it holds that

$$|\{i \mid x_i \in A\}| \text{ is odd} \Leftrightarrow f(x_1, x_2, \dots, x_{2m}) \in B,$$

then B is Θ_2^P -hard.

Proof Sketch: Θ_2^P -Hardness

Concatenation of the construction used to show DP-hardness.



There is some odd i such that $\varphi_i \in \text{SAT}$ and $\varphi_{i+1} \notin \text{SAT}$

\Leftrightarrow

there is a minimal upward covering set that contains d .

Summary of Results

Problem	MC_u, MC_d	MSC_u	MSC_d
Size	NP-complete	NP-complete	NP-complete
Member	Θ_2^P -hard, in Σ_2^P	Θ_2^P -complete	coNP-hard, in Θ_2^P
Member-All	coNP-complete	Θ_2^P -complete	coNP-hard, in Θ_2^P
Unique	coNP-hard, in Σ_2^P	coNP-hard, in Θ_2^P	coNP-hard, in Θ_2^P
Test	coNP-complete	coNP-complete	coNP-complete
Find	not in polynomial time unless $P = NP$		

Thank you for your attention!

The Complexity of Computing Minimal Unidirectional Covering Sets, D. Baumeister, F. Brandt, F. Fischer, J. Hoffmann, and J. Rothe, *Technical Report, July 2009*.