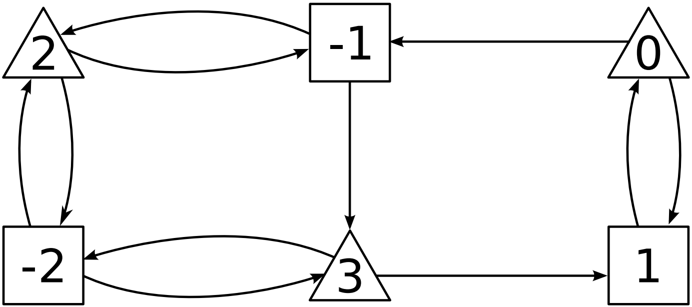
Non-oblivious Strategy Improvement

John Fearnley

University of Warwick

GASICS Meeting 23rd October 2009

Mean-Payoff Games



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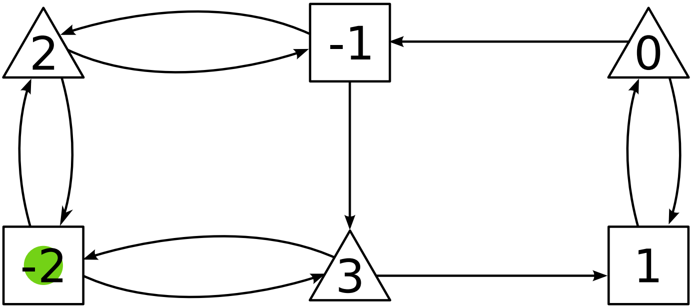
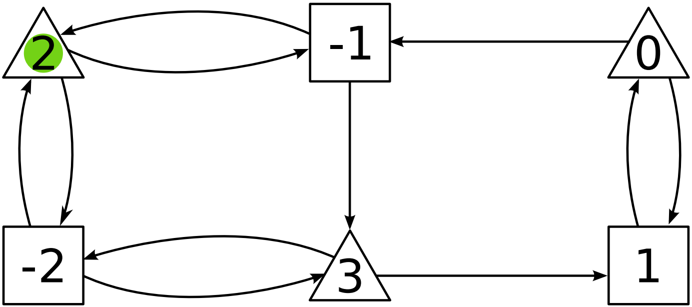


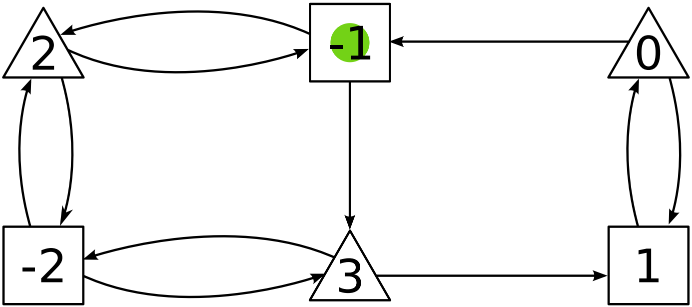
Rewards = -2,

Mean-Payoff Games



Rewards = -2, 2,

Mean-Payoff Games



Rewards = -2, 2, -1,

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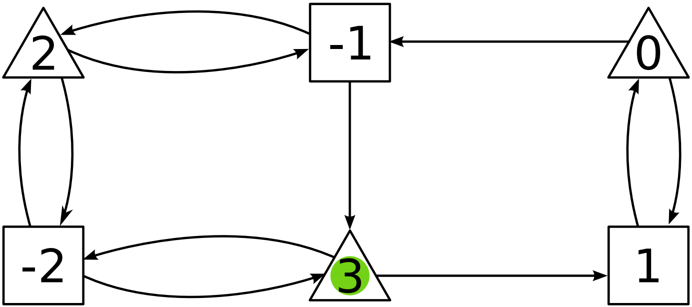

Rewards = -2, 2, -1, 3,

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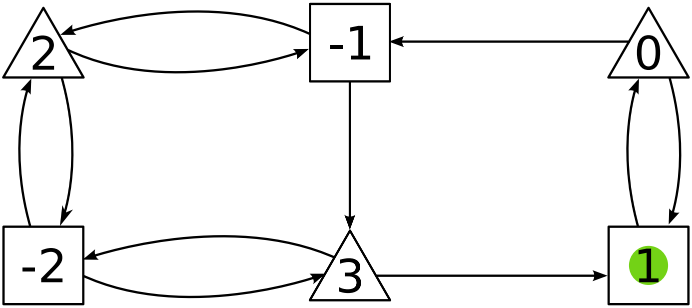

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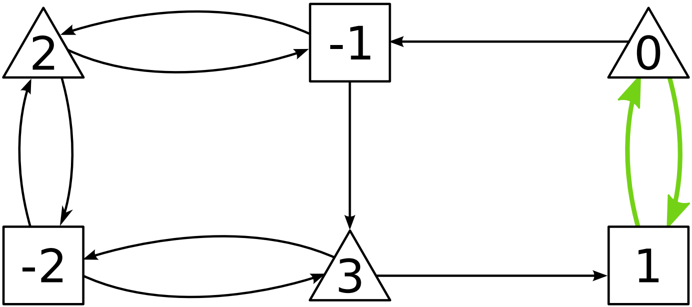

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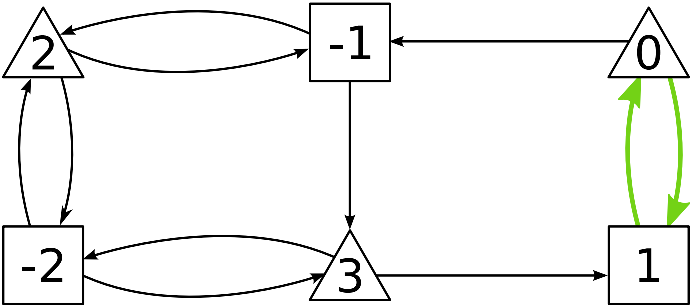

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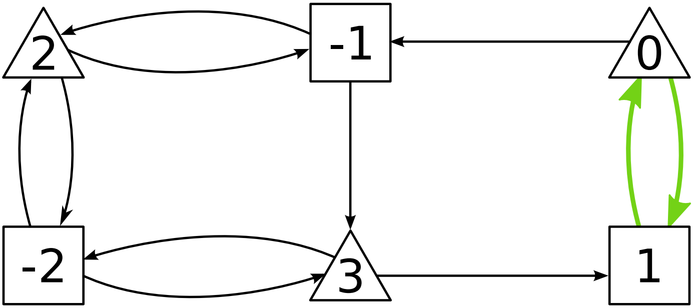

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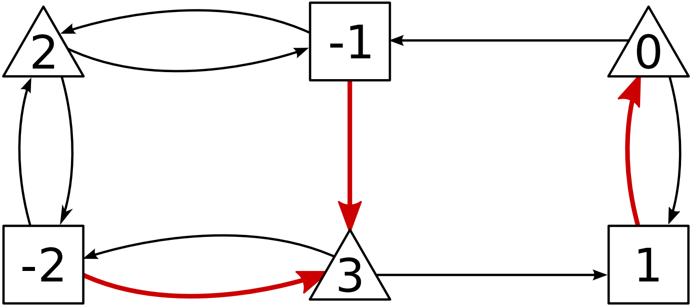
Positional Strategies



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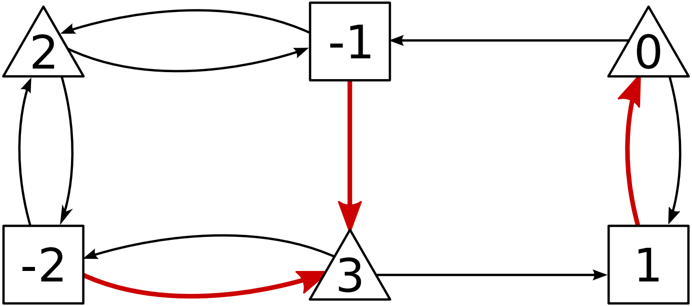

Zero Mean Partition Problem

For each vertex, does Max have a positional strategy that guarantees payoff > 0?

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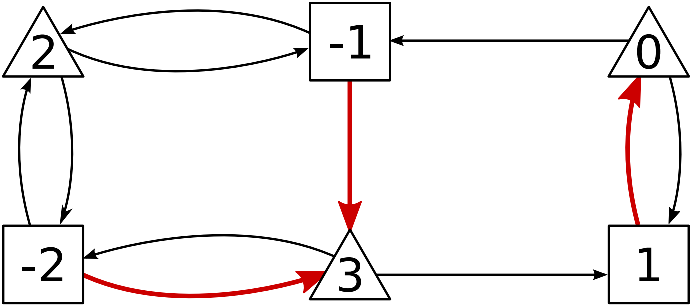

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For each vertex, does Max have a positional strategy that guarantees payoff > 0?

$\Leftrightarrow$

Does Max have a strategy against which every cycle is positive?

Motivation

An algorithm for zero mean partition can be used to find the exact value of each vertex. (Björklund and Vorobyov 2007)

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- Mean payoff games have applications in online scheduling and string comparison problems.
- Parity games can be reduced to mean payoff games.
- Model checking μ-calculus is polynomial time equivalent to the problem of solving a parity game.
- The problem is in NP $\cap$ co-NP but no polynomial time algorithm is known.

Strategy Improvement

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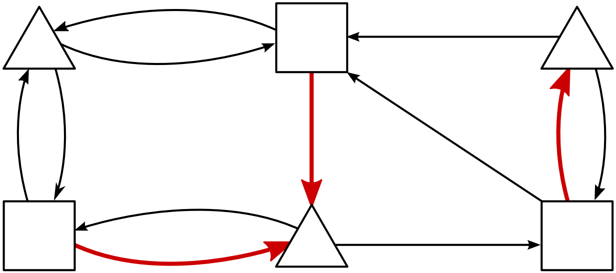

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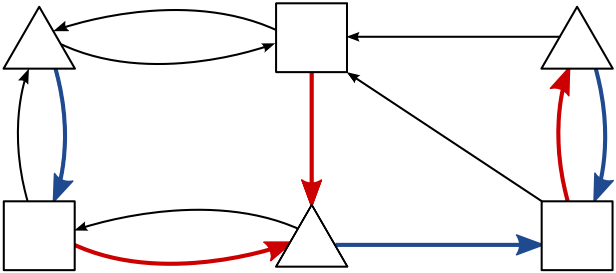

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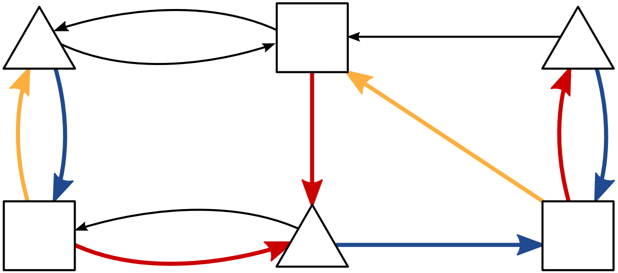

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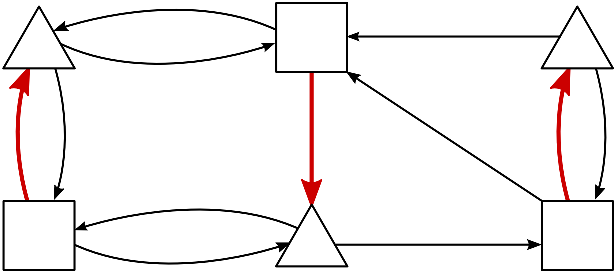

- Pick a strategy for one of the players.
- Compute a best response for the other player.
- Find a set of profitable edges.
- Switch some profitable edges to create an improved strategy.

Strategy Improvement

There are strategy improvement algorithms for:

- Simple stochastic games (Condon 1993)
- Discounted games (Puri 1995)
- Parity games (Vöge and Jurdziński 2000)
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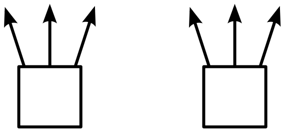

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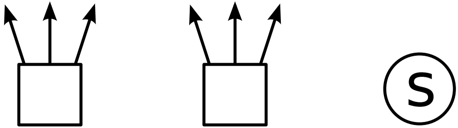

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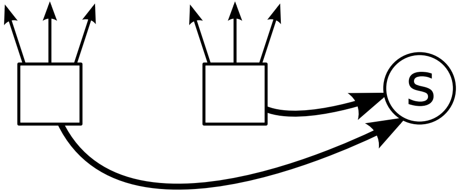

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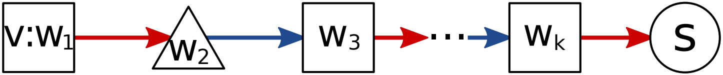

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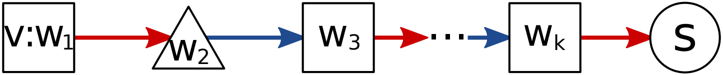

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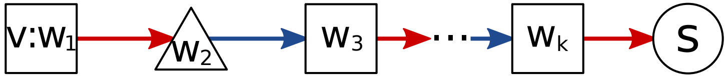

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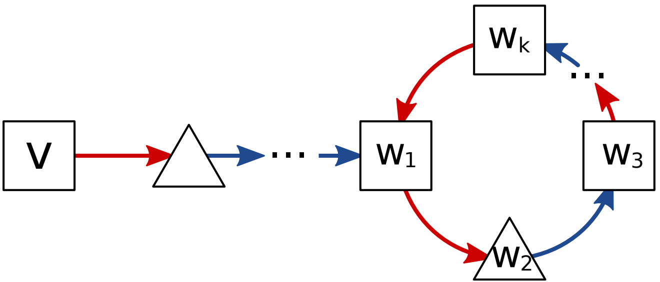

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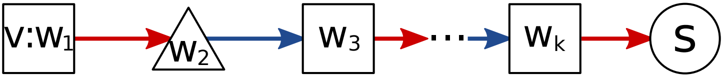

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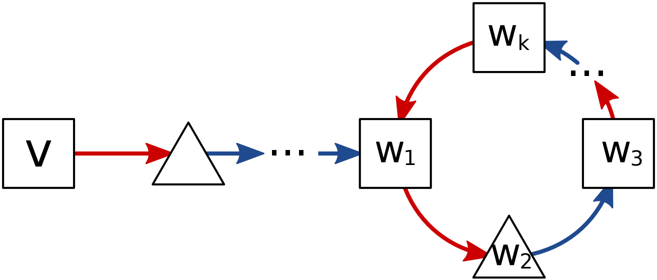

$$\text{Val}^{\sigma,\tau}(v) = \begin{cases} \infty & \text{if } \sum_{i=1}^{k} w_i > 0 \\ -\infty & \text{otherwise} \end{cases}$$

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Proposition (Björklund and Vorobyov 2007)

For every strategy σ the best response $\mathsf{br}(\sigma)$ can be computed in polynomial time.

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Improving Strategies

Strategies will be improved by switching profitable edges.

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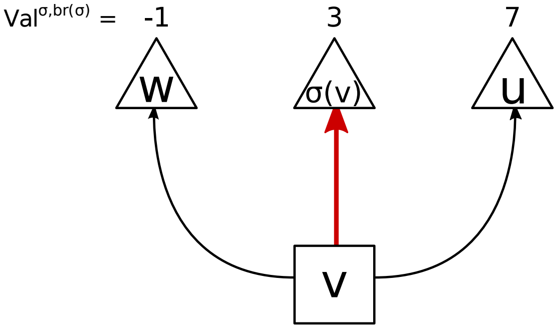

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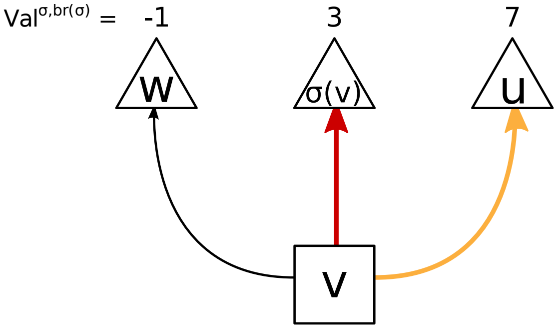

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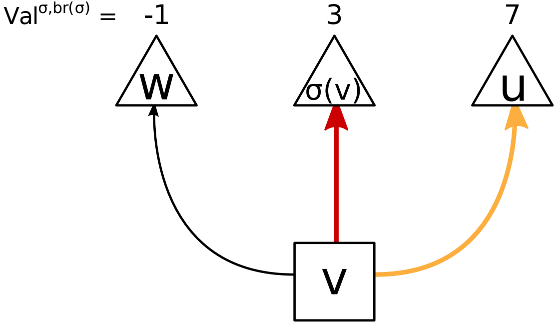

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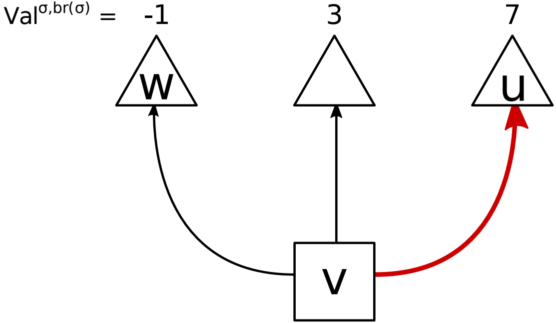

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Let P be some subset of profitable edges in σ, and let σ' be σ with every edge in P switched. We have:

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A strategy with no profitable edges is optimal.

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Strategy improvement algorithms look like this:

while σ has a profitable edge **do**
  Compute the best response $\text{br}(\sigma)$.
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A switching policy picks the subset that is switched.

Switching Policies

Single-vertex switching policies.

- Can take exponential time. (Lebedev 1988)
- A randomized single-vertex policy is subexponential. $O(2^{\sqrt{(n \log n)}})$ (Björklund and Vorobyov 2007)

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Theorem (Friedmann 2009)

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Our answers to these questions come from profitable back edges.

Strategy Trees

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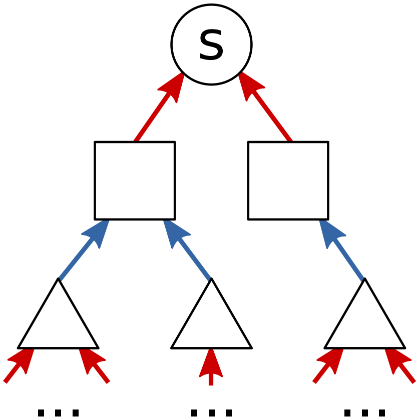
The tree of a Max strategy σ is a tree rooted at the sink whose edges are those chosen by σ and $\mathrm{br}(\sigma)$.



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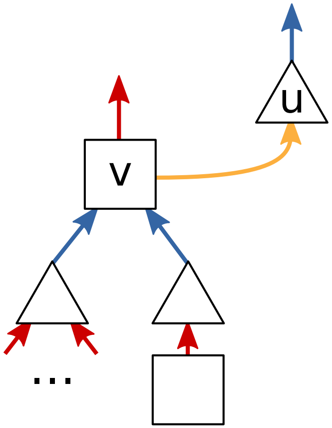

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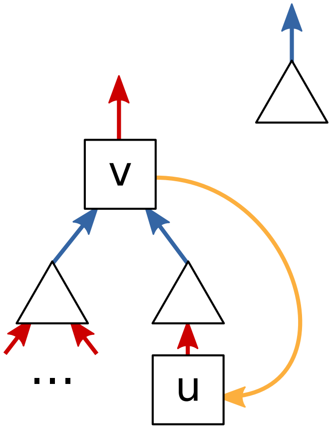
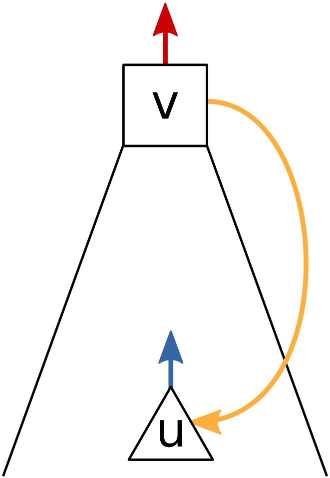


Back Edges



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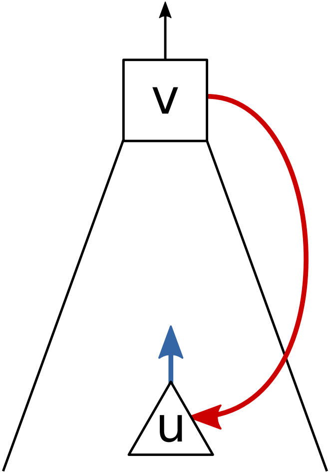

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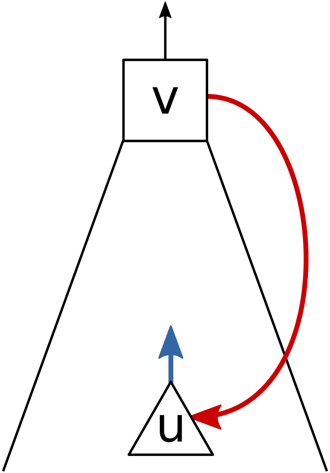

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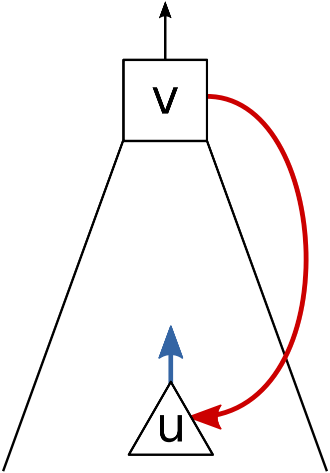

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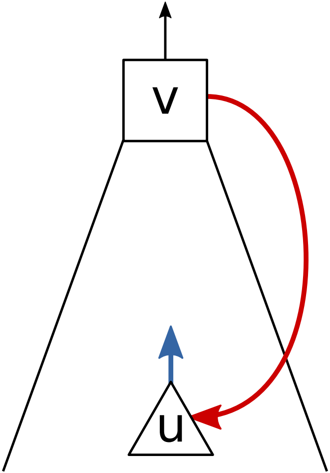
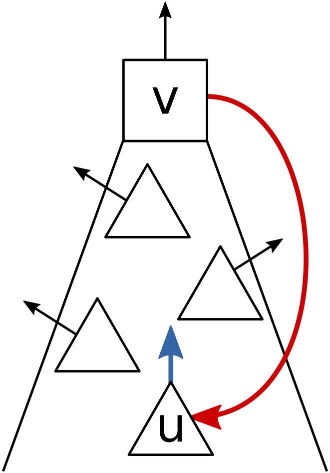
The strategy is winning for the subgame induced by the subtree of v.

Back Edges



How can vertices be reconnected to the sink?

Back Edges



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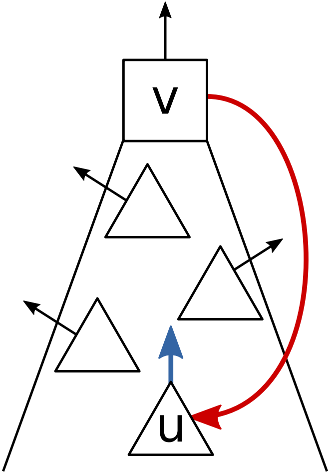
An escape edge is an edge that Min can use to leave the subtree of v.

Back Edges



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An escape edge is an edge that Min can use to leave the subtree of v.

To reconnect the subtree with the sink, the best response must use at least one escape edge.

Remembering Back Edges

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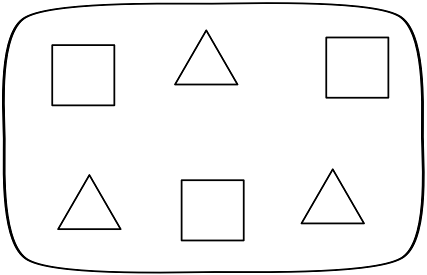
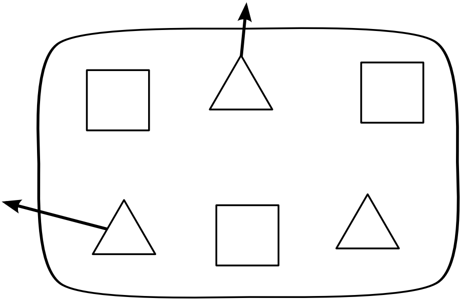


For each profitable back edge we record:

- the set of vertices in the subtree of v.

Remembering Back Edges



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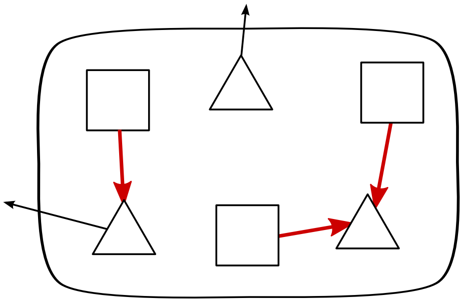

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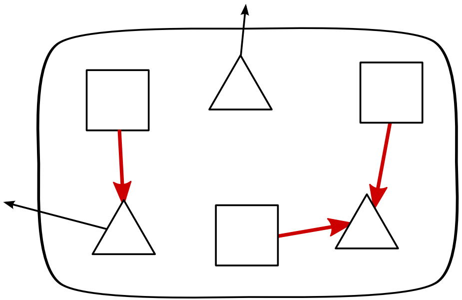

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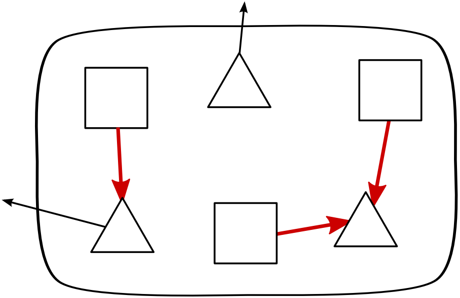

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- the set of vertices in the subtree of v.
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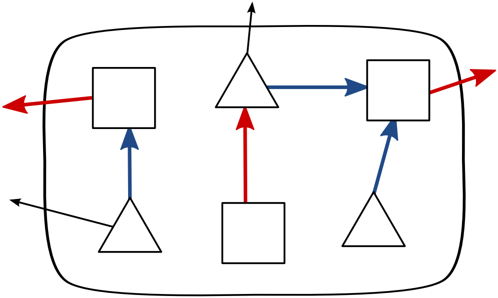
We call this a snare.

Using Snares

Suppose that we have a strategy σ such that $\mathrm{br}(\sigma)$ does not use an escape from the snare.



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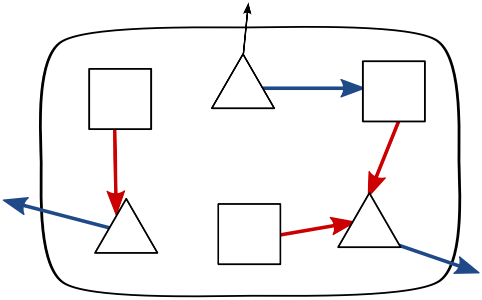

We have a procedure FixSnare which produces a strategy σ' with:

- some escape edge is chosen by $\text{br}(\sigma')$

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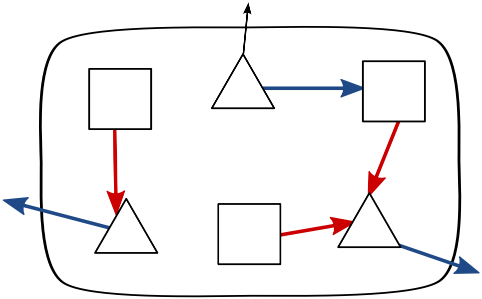

We have a procedure FixSnare which produces a strategy σ' with:

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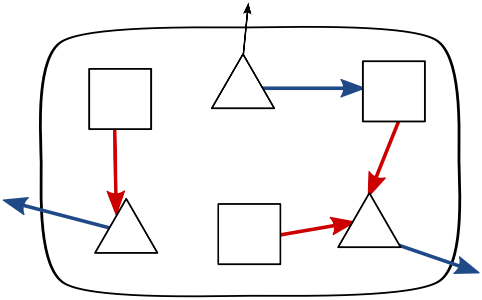

We have a procedure FixSnare which produces a strategy σ' with:

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FixSnare terminates in polynomial time.

How does this help?

What is the advantage of using this procedure?

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FixSnare can produce a strategy that is better than the strategy obtained by switching an optimal subset of profitable edges!

Non-oblivious Strategy Improvement

while σ has a profitable edge **do**

Compute the best response $\mathsf{br}(\sigma)$.
Compute $\mathsf{Val}^{\sigma,\mathsf{br}(\sigma)}(v)$ for every vertex v.
Compute the set of profitable edges in σ.

$\sigma := \sigma$ with some subset of profitable edges switched.

end while

Non-oblivious Strategy Improvement

while σ has a profitable edge **do**

  $\texttt{Snares} := \texttt{Snares} \cup$ snares for all profitable back edges in σ.

  Compute the best response $\text{br}(\sigma)$.

  Compute $\text{Val}^{\sigma,\text{br}(\sigma)}(v)$ for every vertex v.

  Compute the set of profitable edges in σ.

  either

    $\sigma := \sigma$ with some subset of profitable edges switched.

  or

    $\sigma :=$ FixSnare(σ, S) for some S in $\texttt{Snares}$.

end while

Non-oblivious Strategy Improvement

Friedmann's examples force oblivious switching policies to switch an exponential number of profitable back edges.

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Theorem

Non-oblivious strategy improvement terminates in polynomial time on Friedmann's examples.

Conclusions

Non-oblivious strategy improvement is not vulnerable to techniques used to show exponential time behaviour for oblivious strategy improvement.

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Future Work

- Prove good complexity bounds for non-oblivious strategy improvement...

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Future Work

- Prove good complexity bounds for non-oblivious strategy improvement...
- or find an exponential time example.