Probabilistic Dependence Logic

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Probabilistic Dependence Logic

- Logics of Imperfect Information
- Hodges' semantics
- Probabilistic Dependence Logic
- Behavioral equilibria and mixed equilibria: a problematic example

Probabilistic Dependence Logic

Game Theoretic semantics for First Order Logic:

Positions = (ψ, s) $(FV(\psi) \subseteq Dom(s))$

Two players, $\forall$**belard** and $\exists$**loise**

Starting position: $(\phi, \emptyset)$

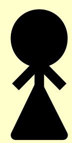
Probabilistic Dependence Logic

Position	***What happens***
$(\psi \vee \theta, s')$	chooses (ψ, s') or (θ, s')
$(\exists x\ \psi, s')$ r	chooses *m*, next = $(\psi, s'[m/x])$
$(\neg\psi, s')$	and switch roles
(ψ, s'), ψ atomic	wins if M, s' |= ψ (otherwise wins)

M |= ϕ iff has a winning strategy for $(\phi, \emptyset)$

Probabilistic Dependence Logic

For logics of imperfect information, add the rule

Position	***What happens***
$((\exists x \backslash W)\ \psi, s')$	 chooses m, next = $(\psi, s'[m/x])$

A strategy τ is *uniform* iff

$$s \equiv_W s',\ \tau(((\exists x \backslash W)\ \psi, s) = (\psi, s[m/x])$$

$$\Downarrow$$

$$\tau(((\exists x \backslash W)\ \psi, s') = (\psi, s'[m/x])$$

Probabilistic Dependence Logic

M |= ϕ iff  has an uniform w. s. for (ϕ, $\emptyset$)

Not all formulas are determined:

$$\forall x\ (\exists y \backslash \{\ \})(x=y)$$

Probabilistic Dependence Logic

$M \models \phi$ iff  has an uniform w. s. for $(\phi, \emptyset)$

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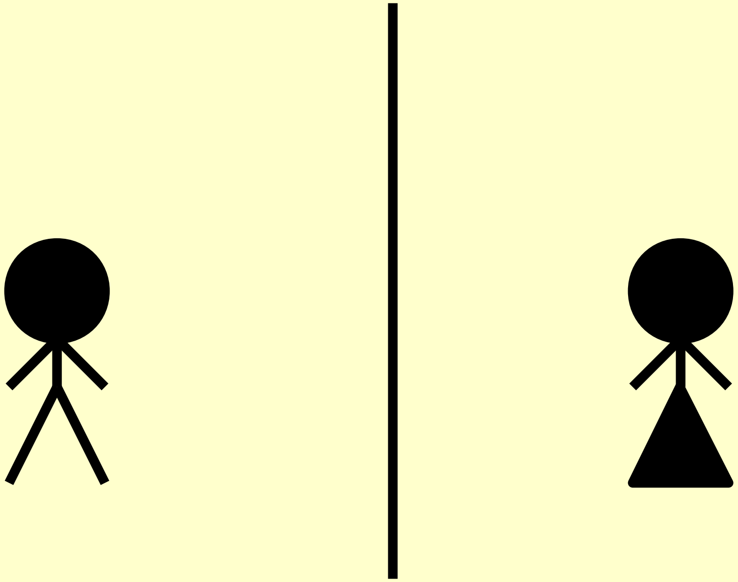

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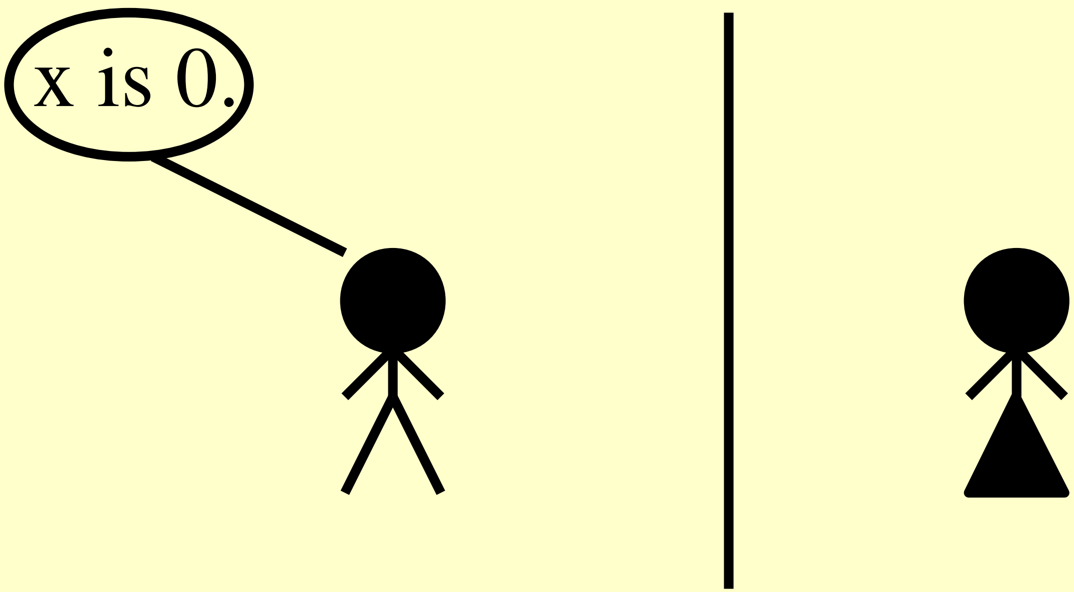

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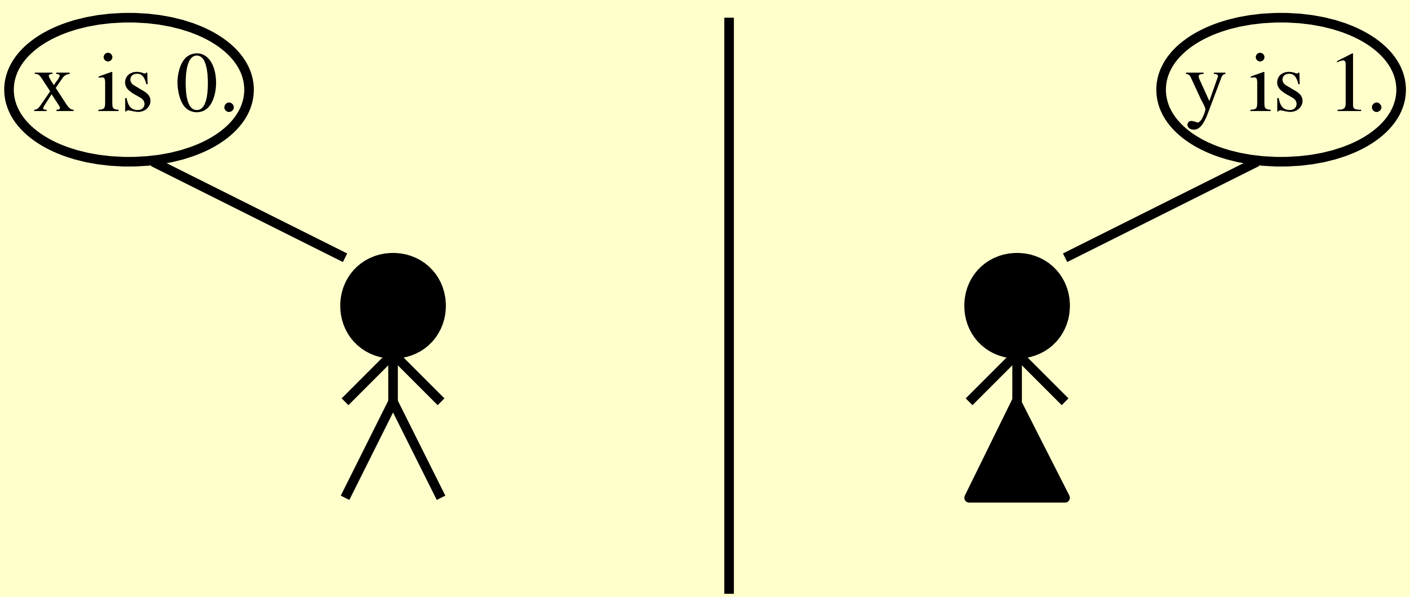

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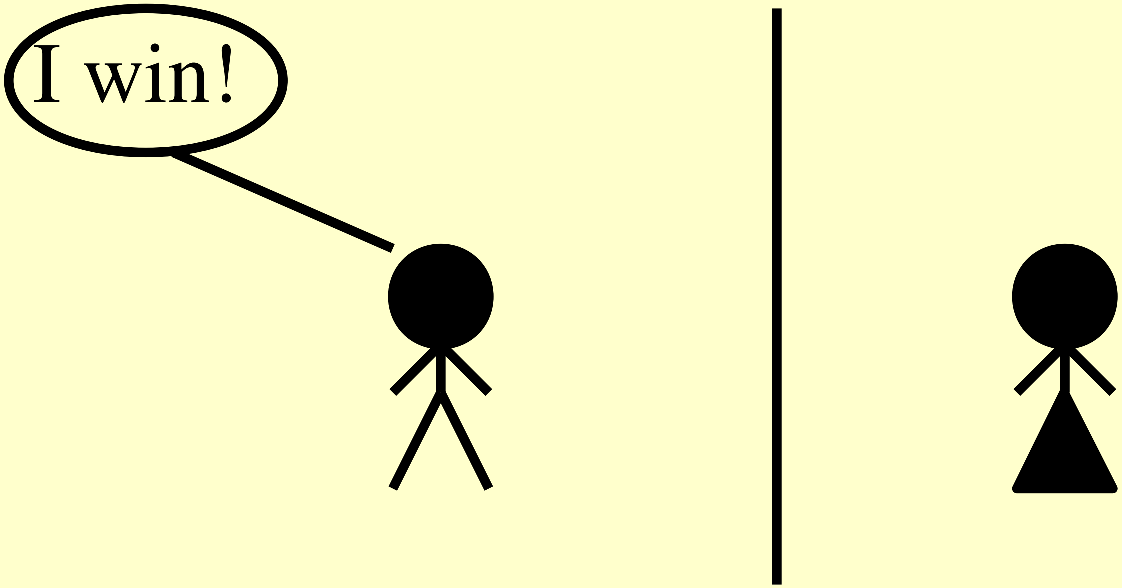

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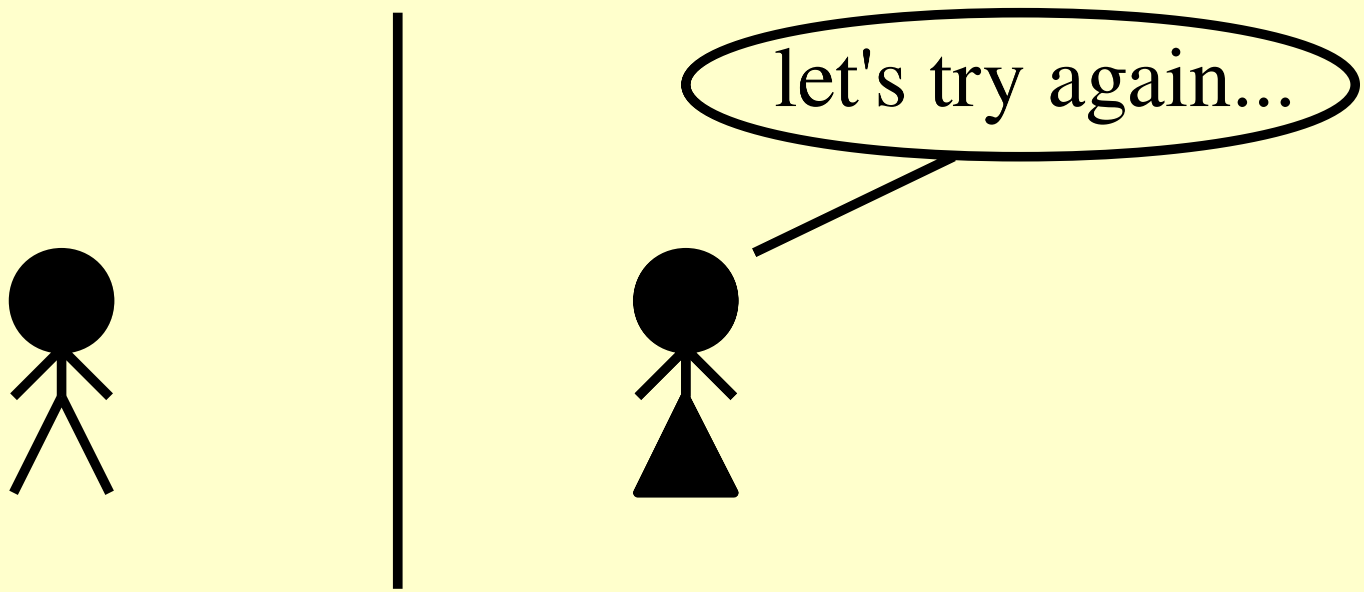

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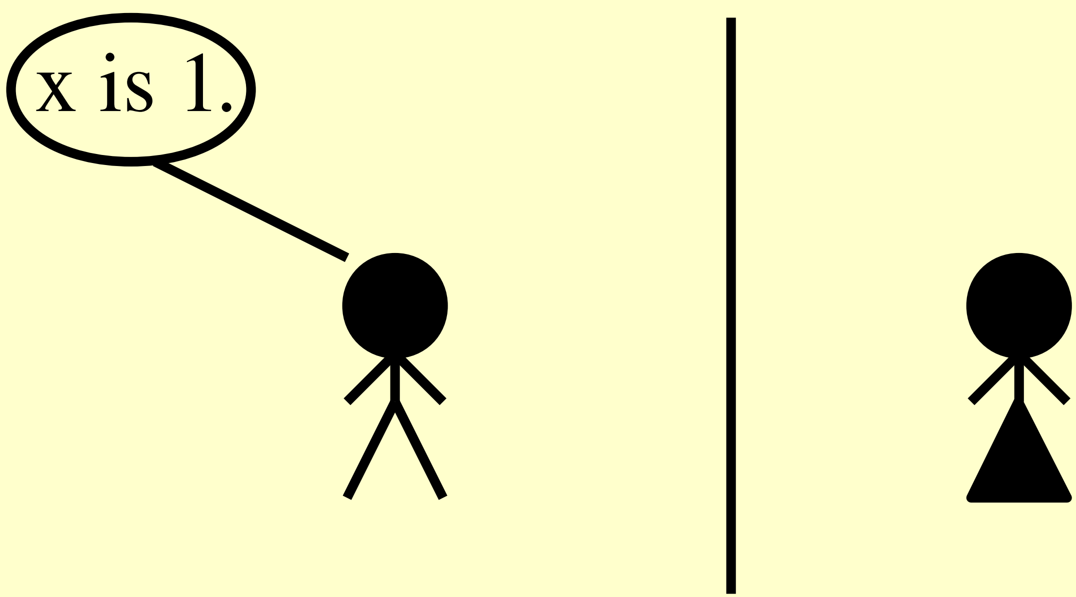

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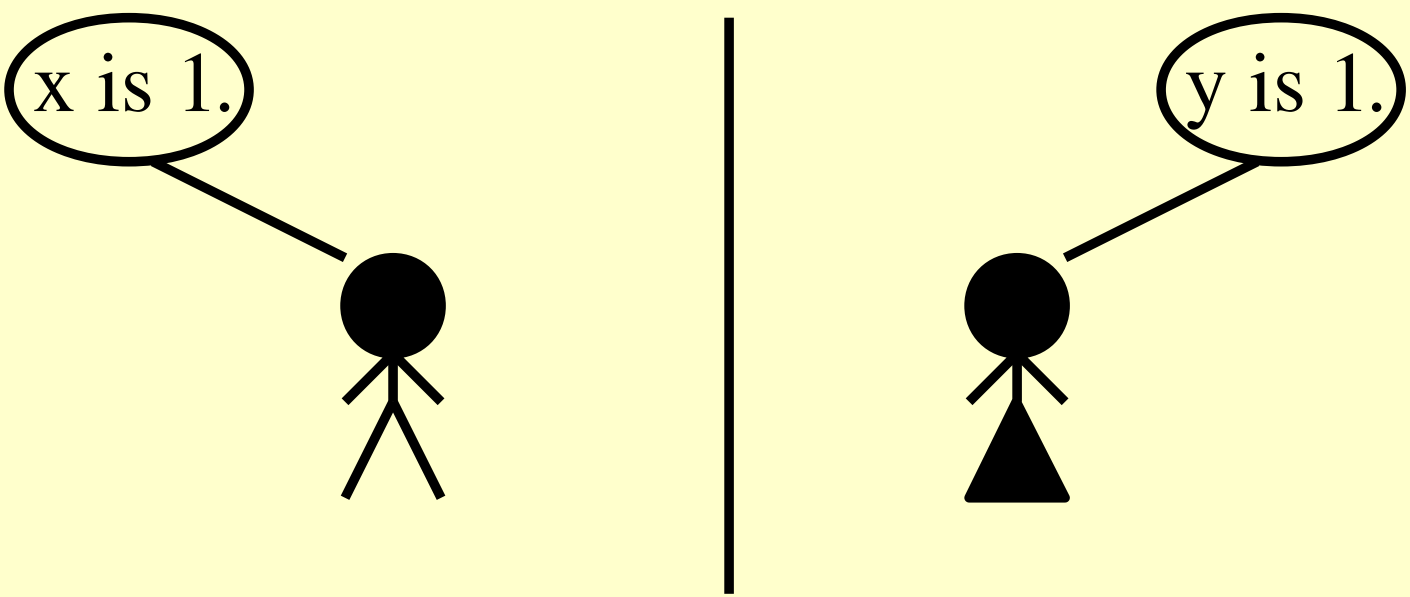

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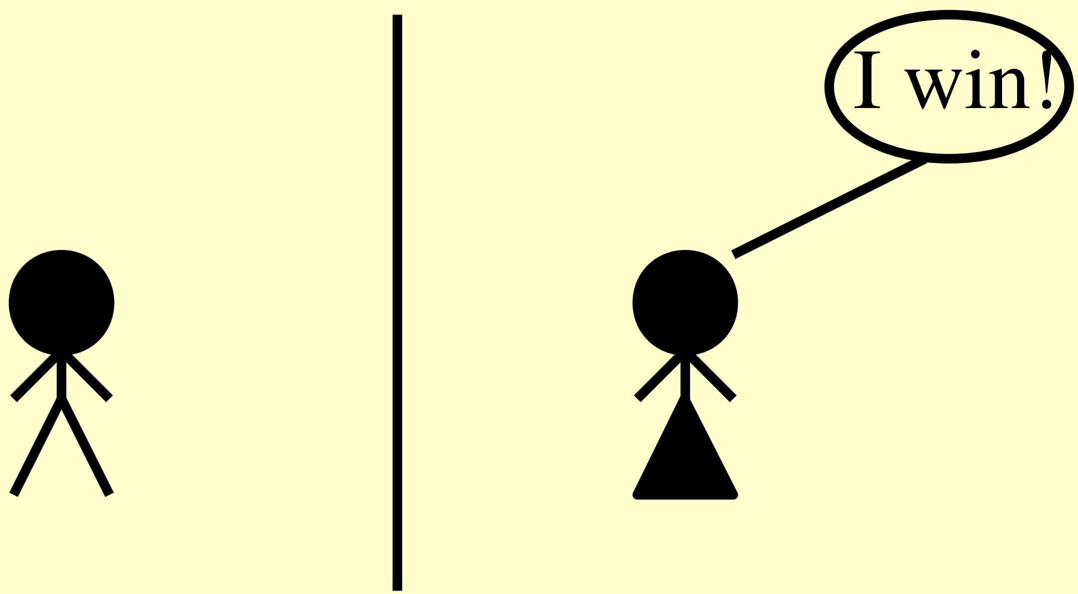
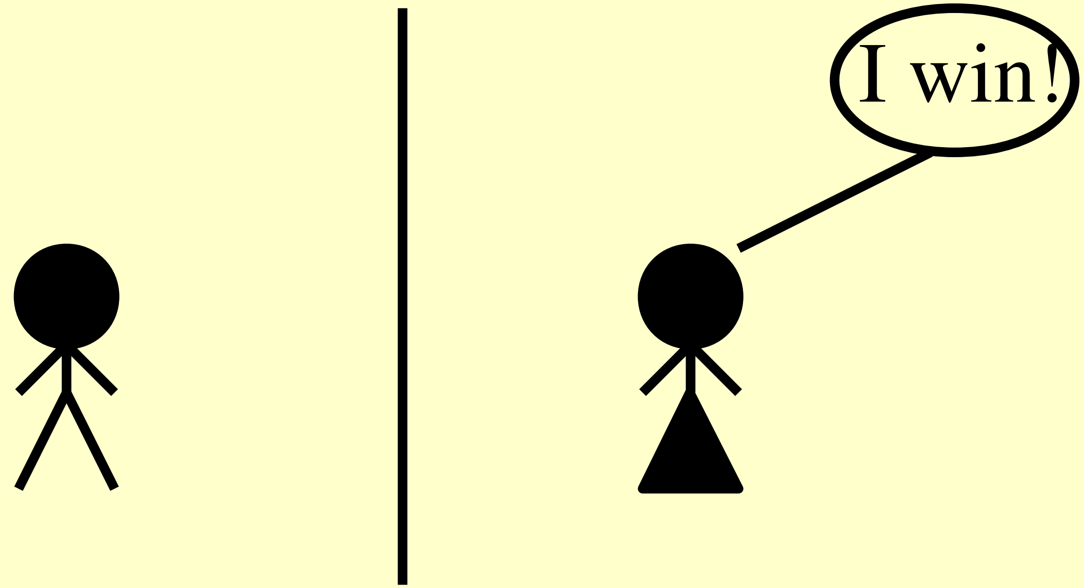


Probabilistic Dependence Logic

Neither player has a uniform winning strategy;
The formula is neither true nor false.

Not all formulas are determined:

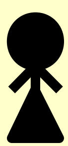
$$\forall x\ (\exists y \backslash \{\ \})(x=y)$$



Probabilistic Dependence Logic

Hodges (1997) developed a compositional semantics for logics of imperfect information.

Teams = sets of assignments **X** (= sets of partial plays reaching some ψ, $FV(\psi) \subseteq Dom(\mathbf{X})$);

$\mathbf{X} \models^{+} \psi$ iff  has a uniform winning strategy for all partial plays in X;

Supplementation: $X[F/x] = \{s[F(s)/x] : s \in x\}$;

Duplication: $X[M/x] = \{s[m/x] : s \in x\}$.

Probabilistic Dependence Logic

- $\mathbf{X} \models^{+} \phi$, ϕ atomic iff $s \models \phi$ for all $s \in \mathbf{X}$;
- $\mathbf{X} \models^{-} \phi$, ϕ atomic iff $s \models \phi$ for all $s \in \mathbf{X}$;
- $\mathbf{X} \models^{+} \phi \vee \psi$ iff $\mathbf{X} \subseteq \mathbf{Y} \cup \mathbf{Z}$, $\mathbf{Y} \models^{+} \phi$, $\mathbf{Z} \models^{+} \psi$;
- $\mathbf{X} \models^{-} \phi \vee \psi$ iff $\mathbf{X} \models^{-} \phi$ and $\mathbf{X} \models^{-} \psi$;
- $\mathbf{X} \models^{+} (\exists x \setminus W)\ \psi$ iff $\mathbf{X[F/x]} \models^{+} \phi$, where
  $s \equiv_W s' \Rightarrow \mathbf{F}(s) = \mathbf{F}(s')$;
- $\mathbf{X} \models^{-} (\exists x \setminus W)\ \psi$ iff $\mathbf{X[M/x]} \models^{-} \phi$;
- $\mathbf{X} \models^{+} \neg\phi$ iff $\mathbf{X} \models^{-} \phi$, $\mathbf{X} \models^{-} \neg\phi$ iff $\mathbf{X} \models^{+} \phi$.

Probabilistic Dependence Logic

$\{\emptyset\} \models^{+} \phi$ iff  has a uniform w.s. in the game for ϕ, that is, iff $\mathbf{M} \models \phi$ in the game semantics.

$\{\emptyset\} \models^{-} \phi$ iff  has a uniform w.s. in the game for ϕ, that is, iff $\mathbf{M} \models \neg\phi$ in the game semantics.

Probabilistic Dependence Logic

$\{\emptyset\}$ $|=^{+}$ ϕ iff  has a uniform w.s. in the game for ϕ, that is, iff **M** $|=$ ϕ in the game semantics.

$\{\emptyset\}$ $|=^{-}$ ϕ iff  has a uniform w.s. in the game for ϕ, that is, iff **M** $|=$ $\neg\phi$ in the game semantics.

But what about undetermined sentences?

Are all of them undetermined in the same way?

Probabilistic Dependence Logic

No! Some sentences are “almost false”, others are “almost true”...

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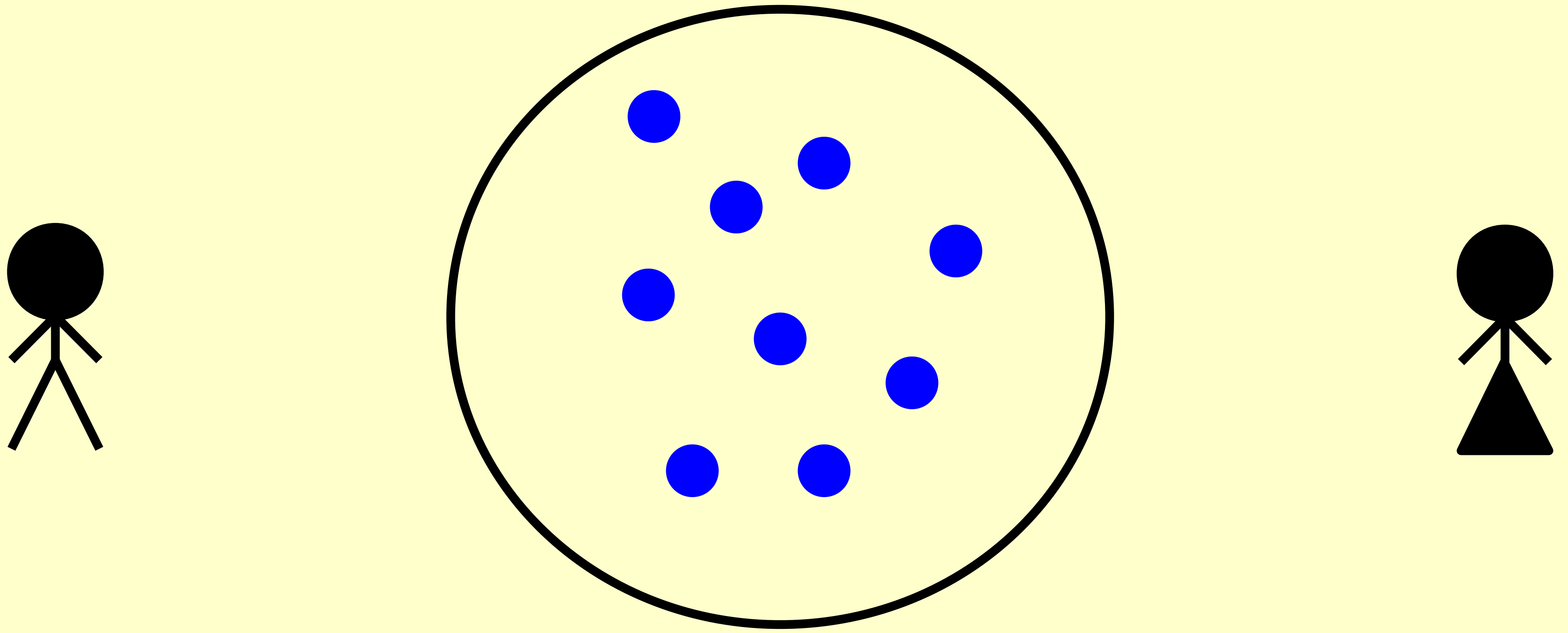

Probabilistic Dependence Logic

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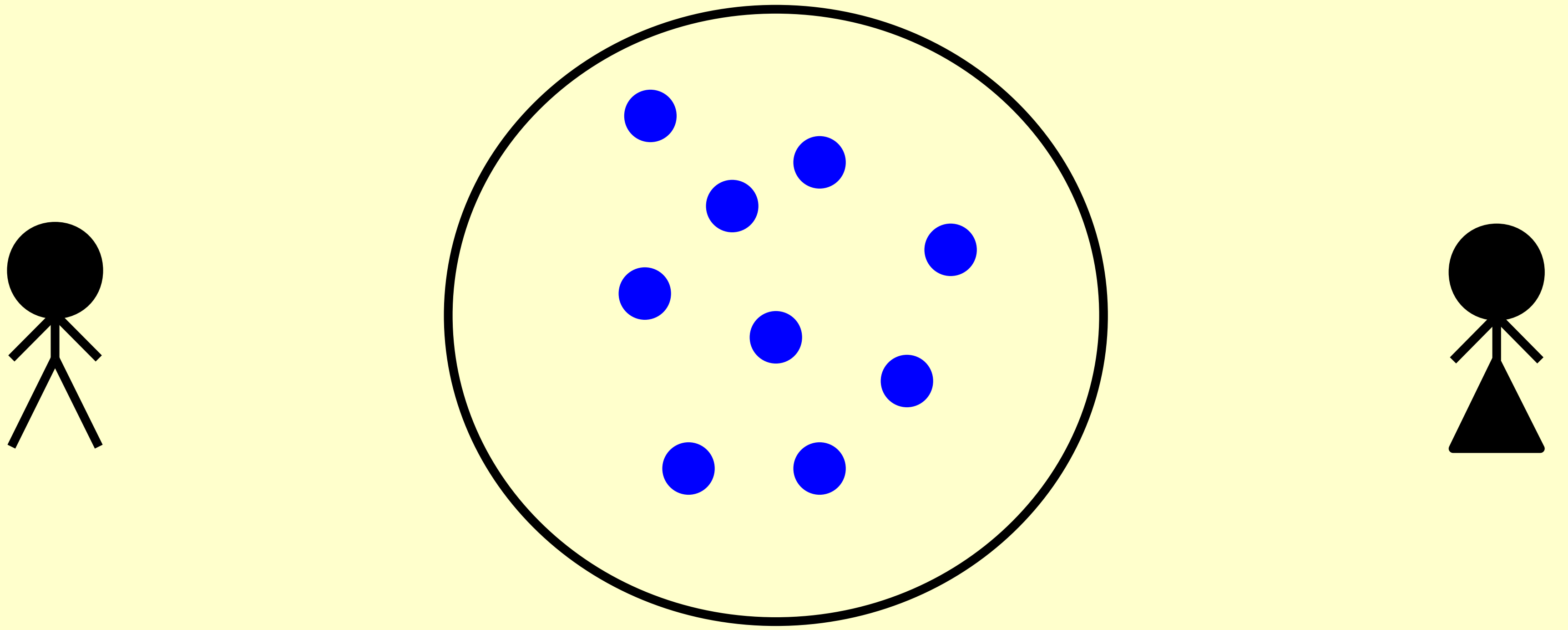
$\forall x\ (\exists y \backslash \{\ \})(x=y)$



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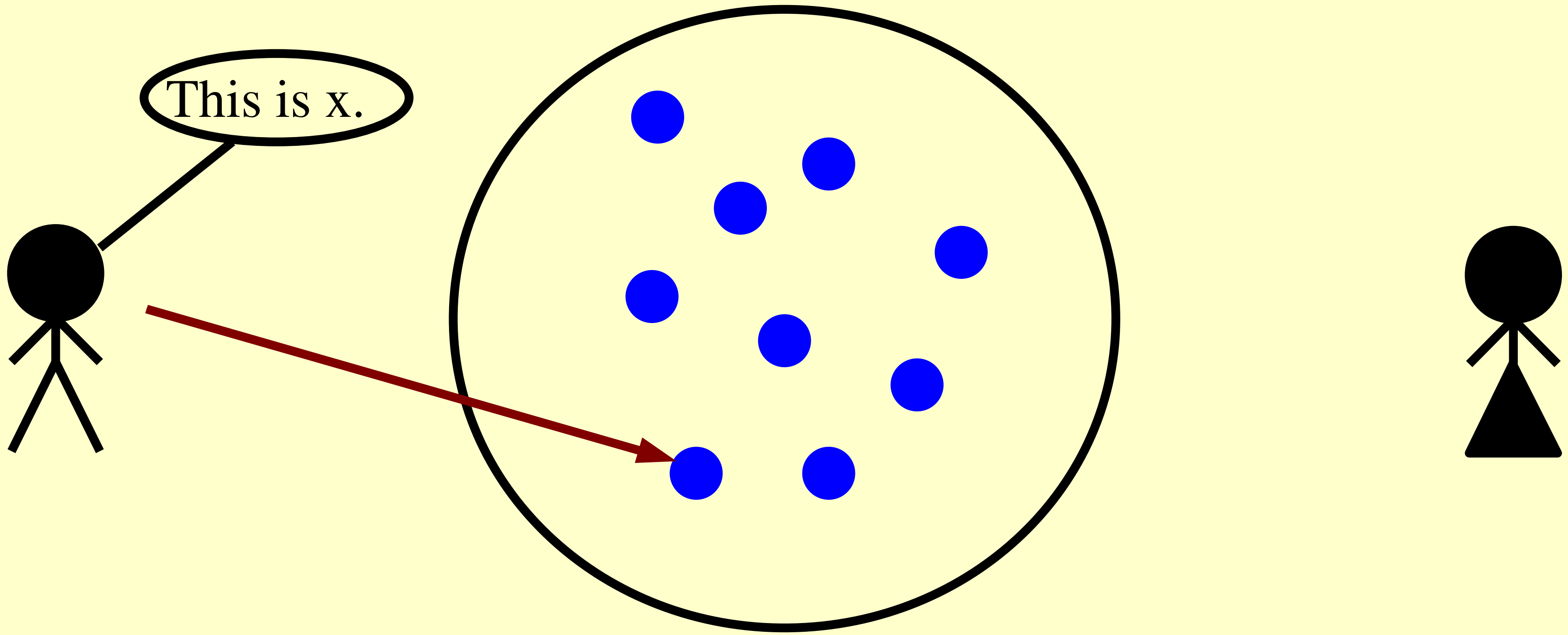

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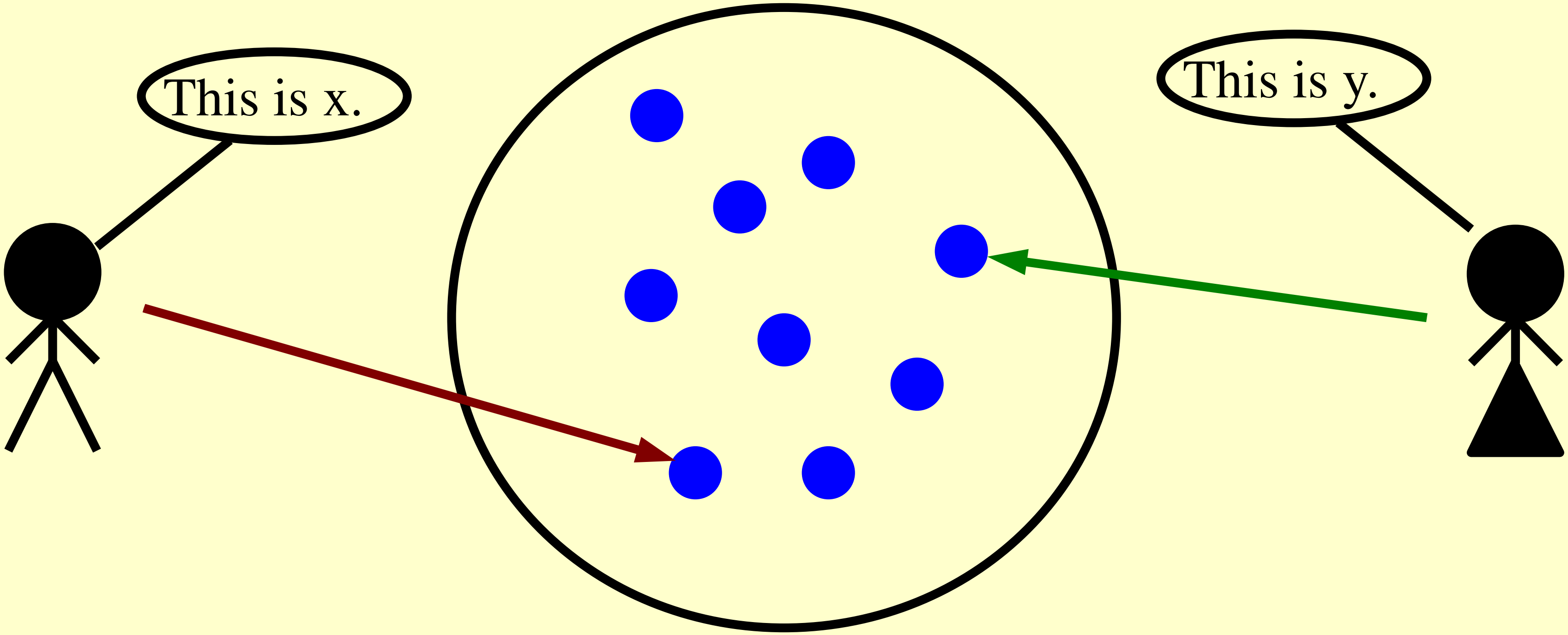

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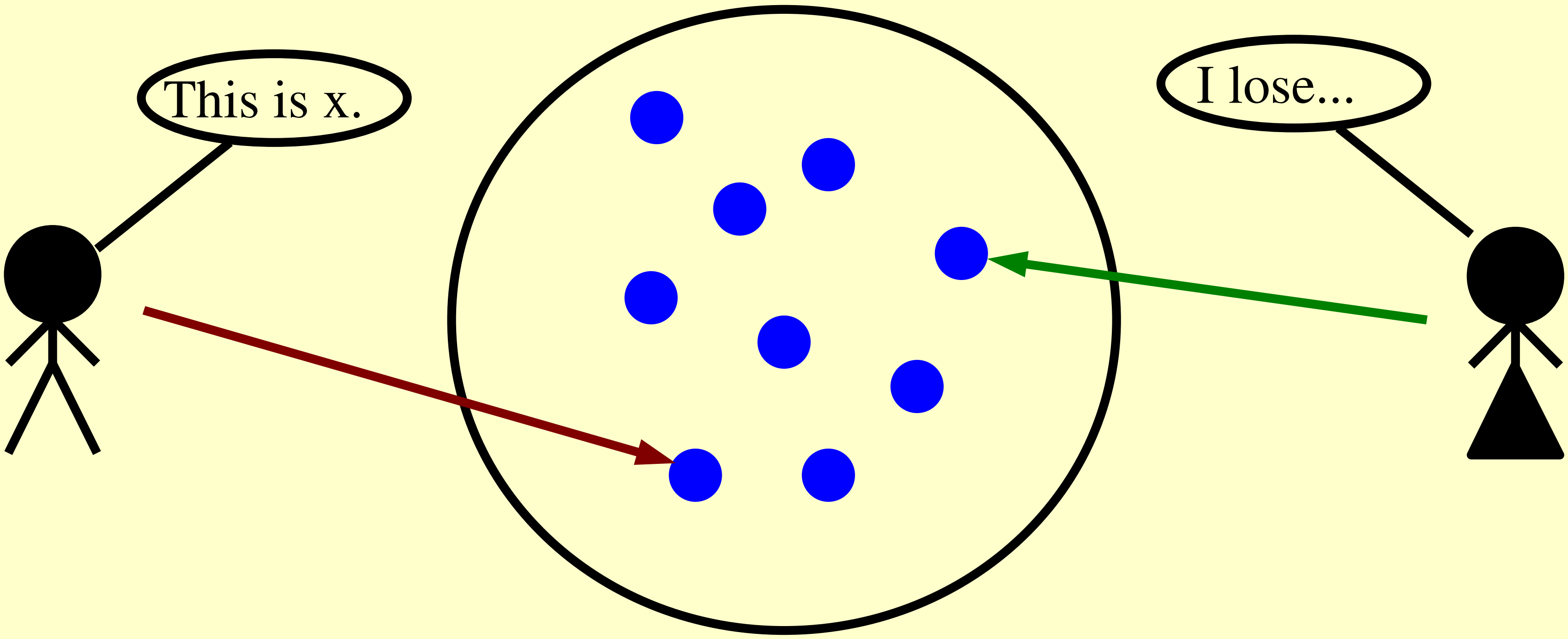

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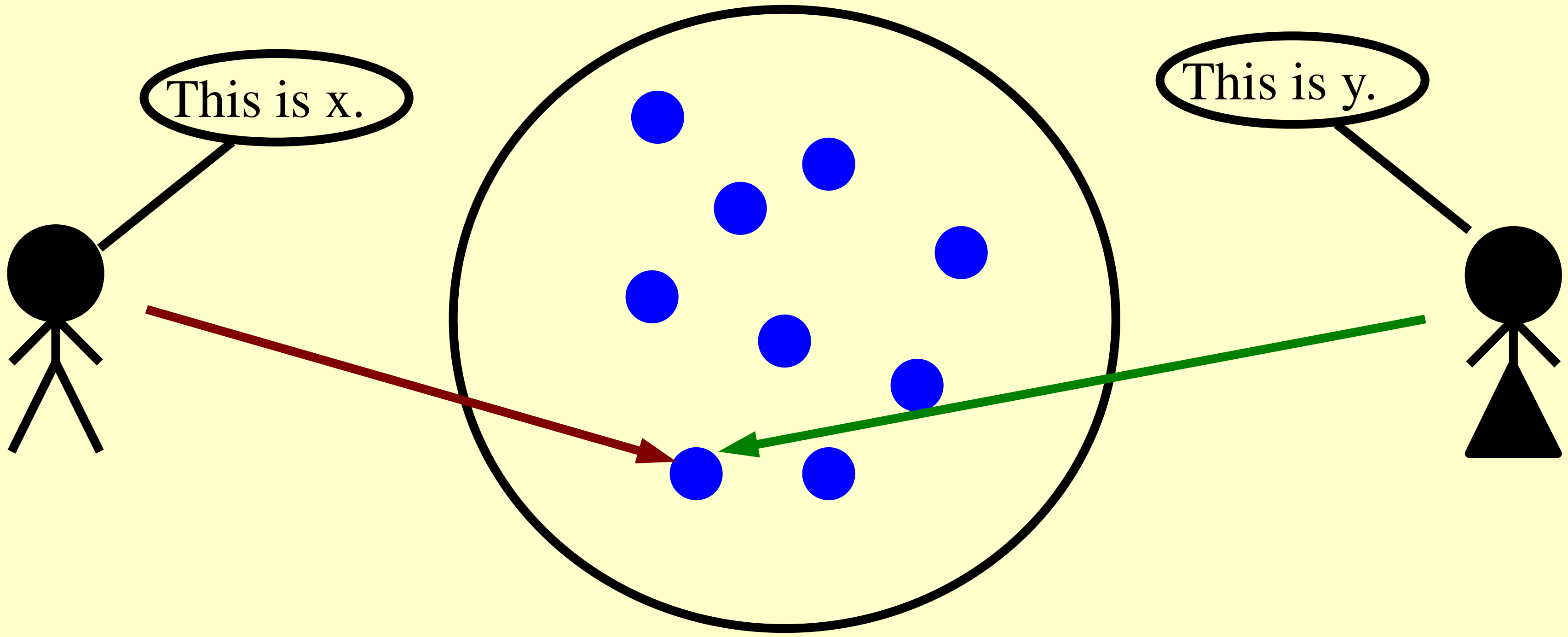

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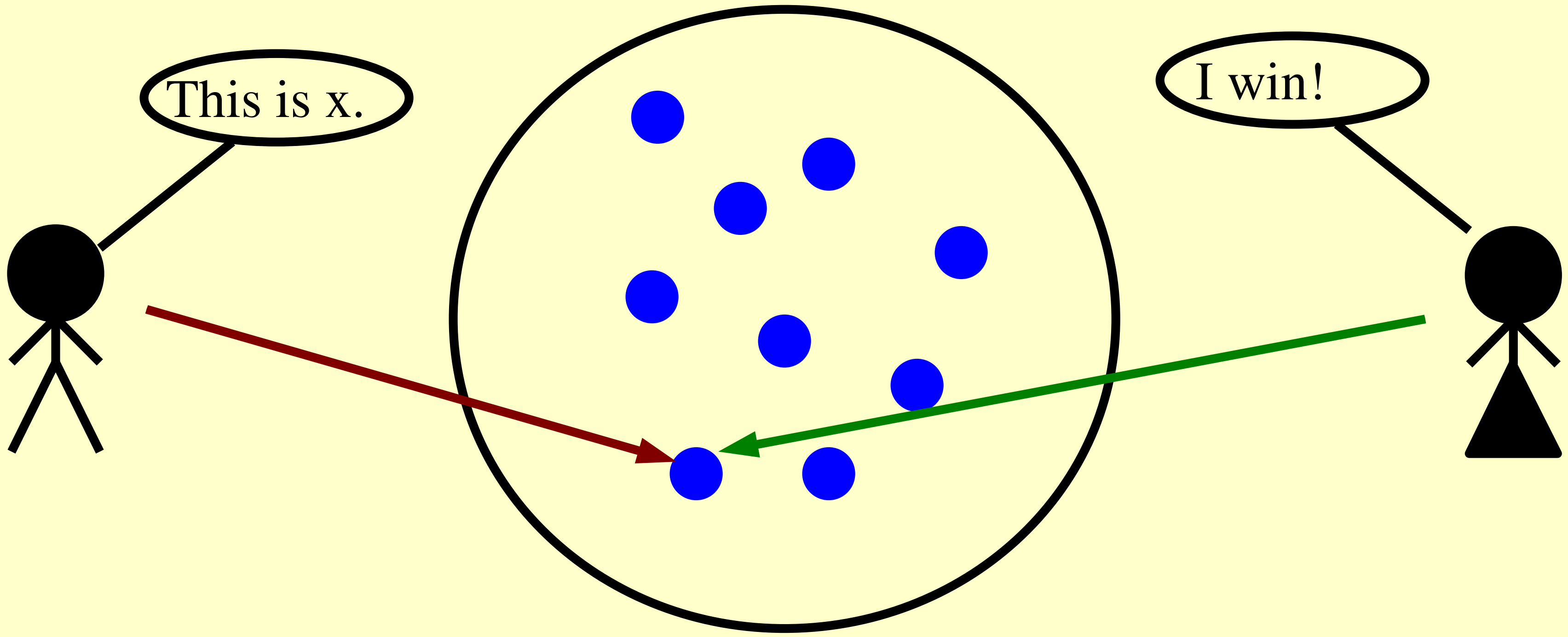

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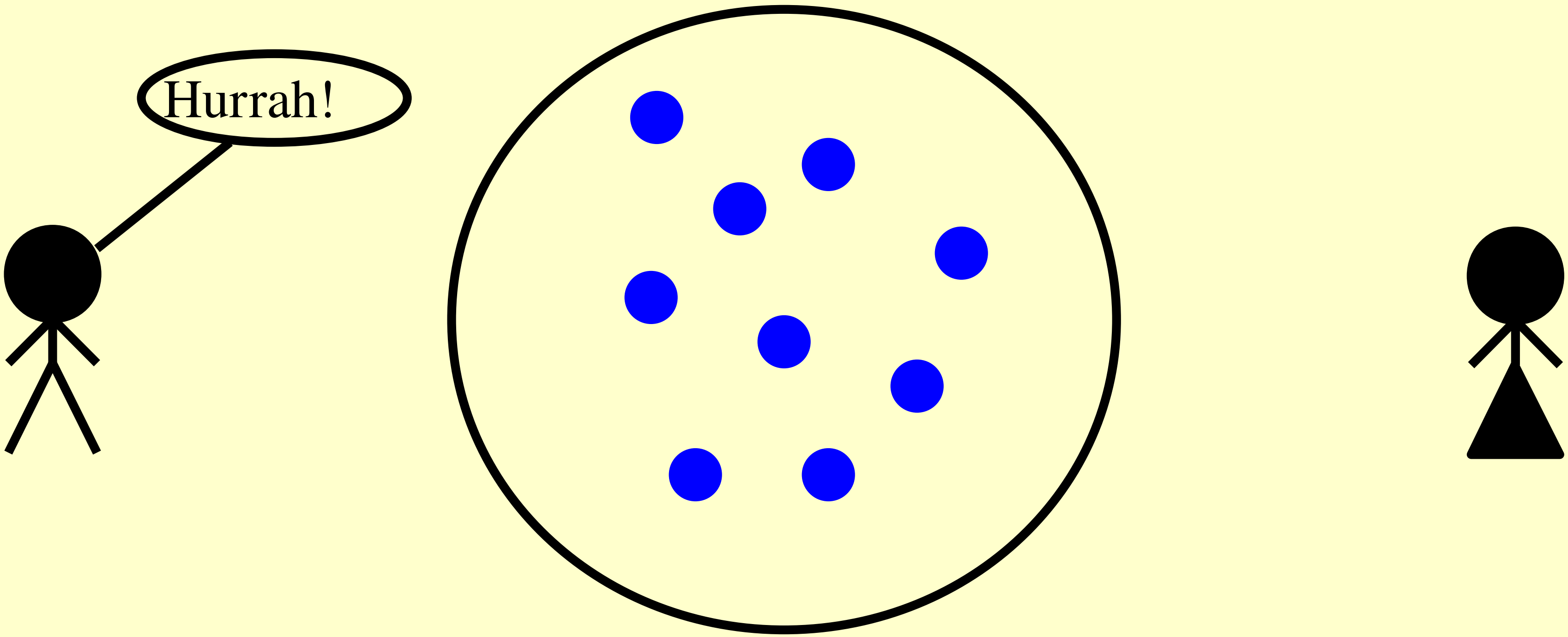

Abelard can wins with high probability: almost false

Probabilistic Dependence Logic

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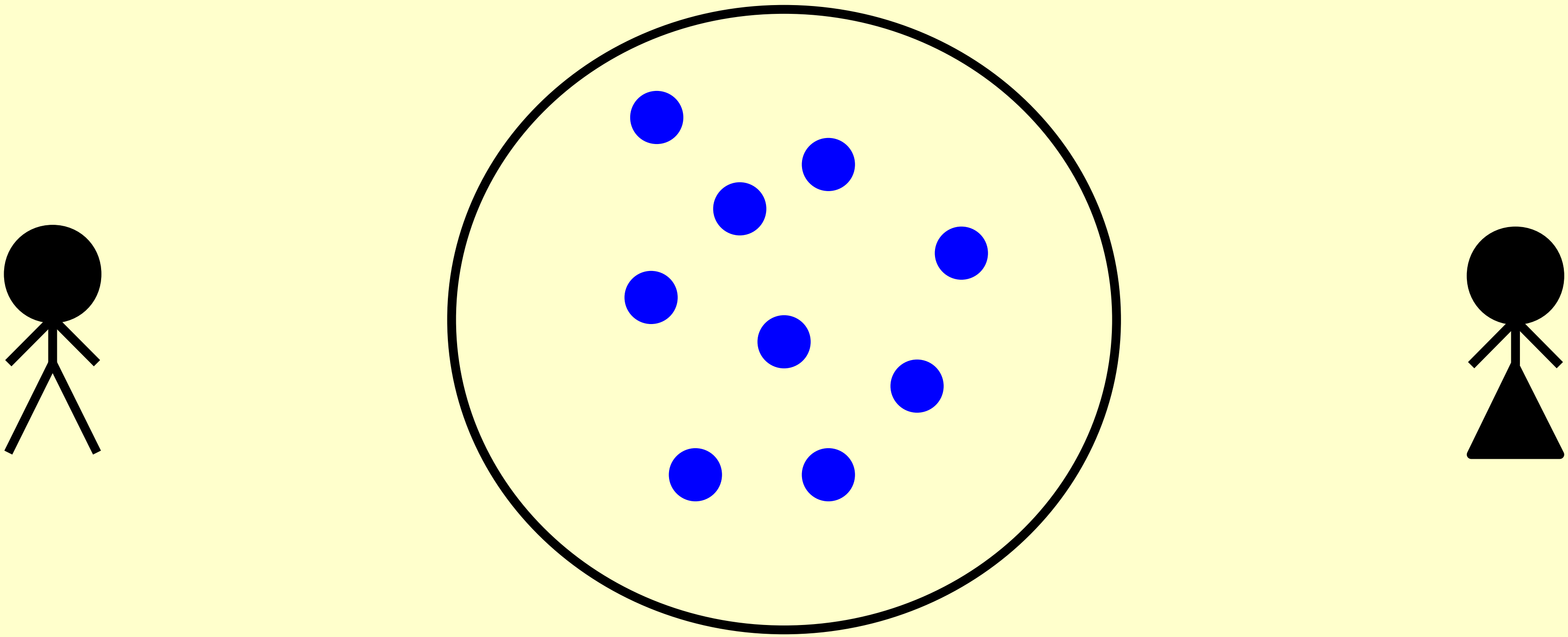
$$\forall x\ (\exists y \backslash \{\ \})(x \neq y)$$



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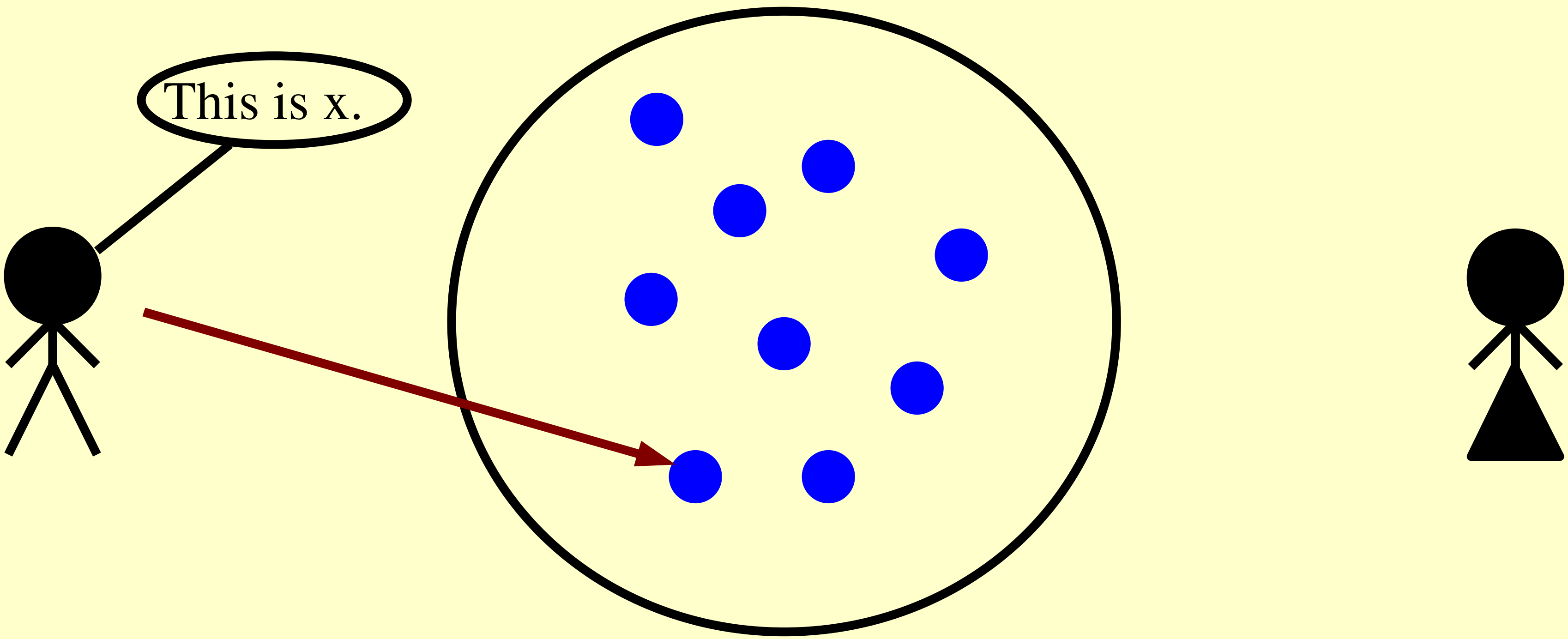

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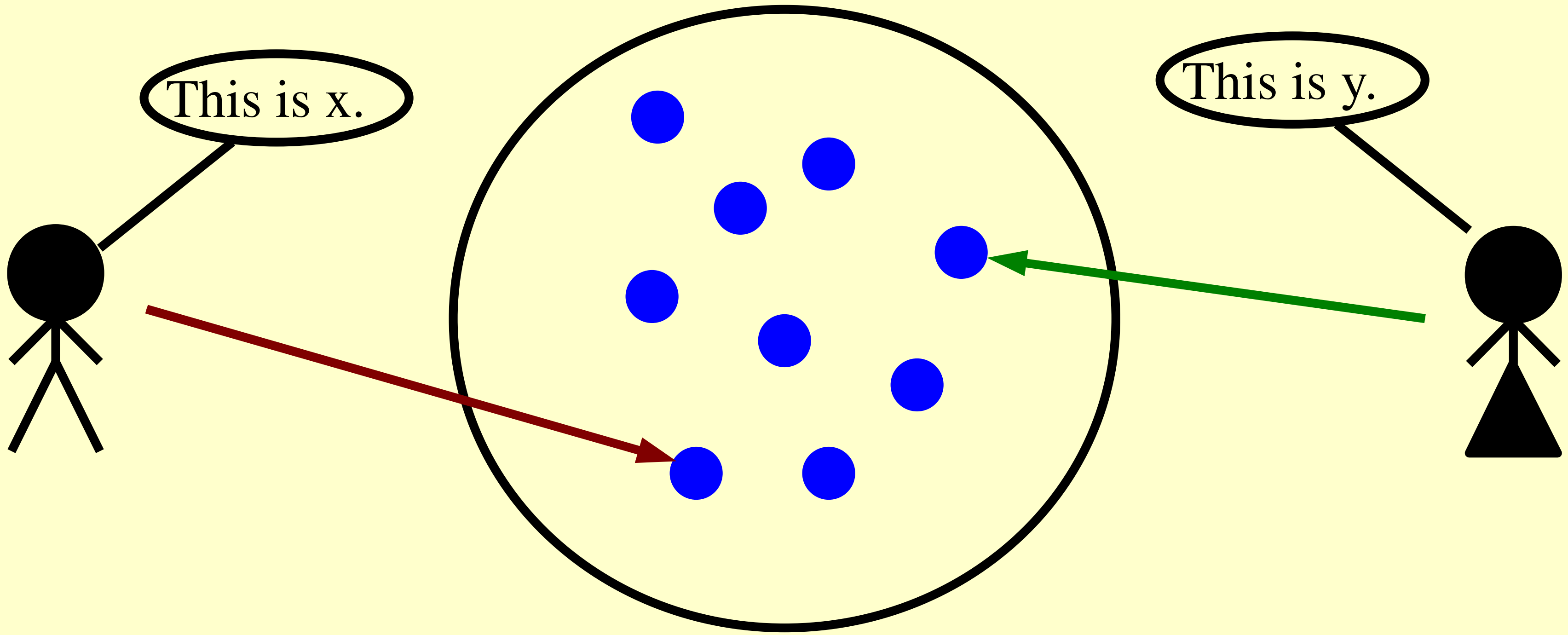

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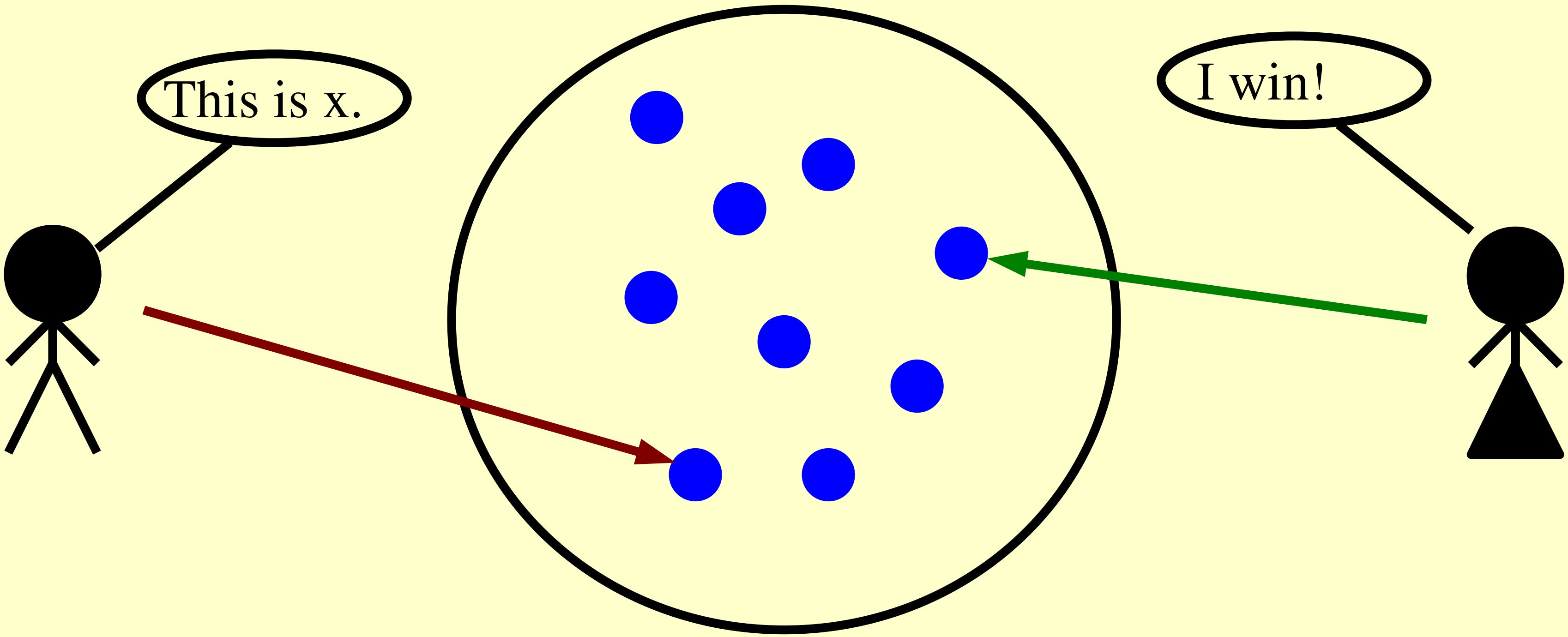

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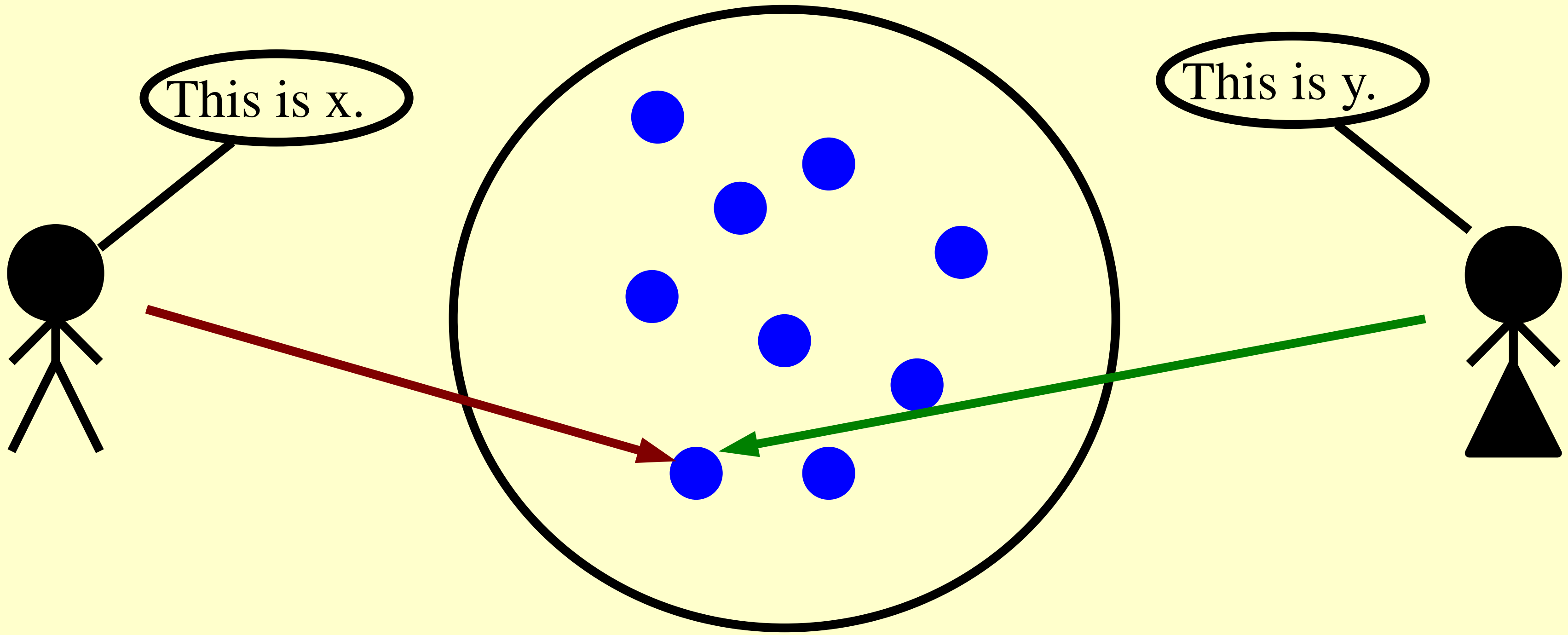
∀x (∃y\{ })(x≠y)



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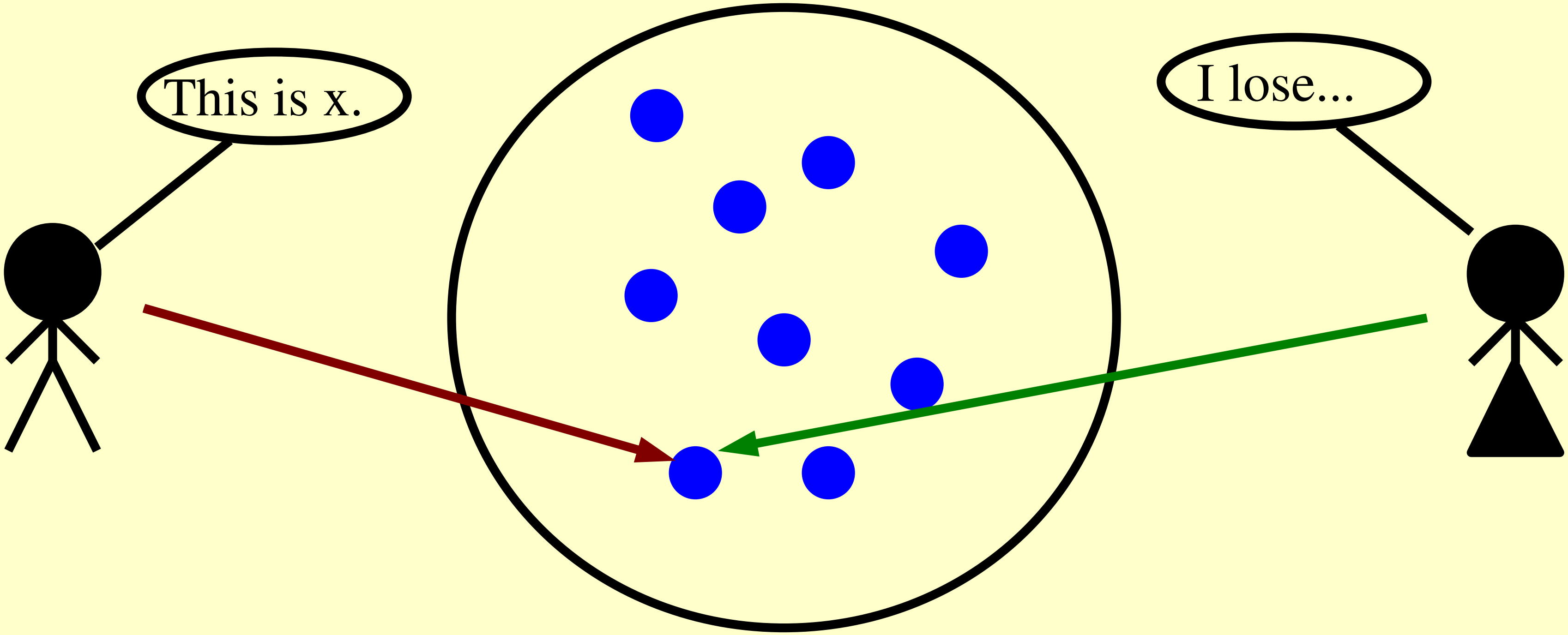

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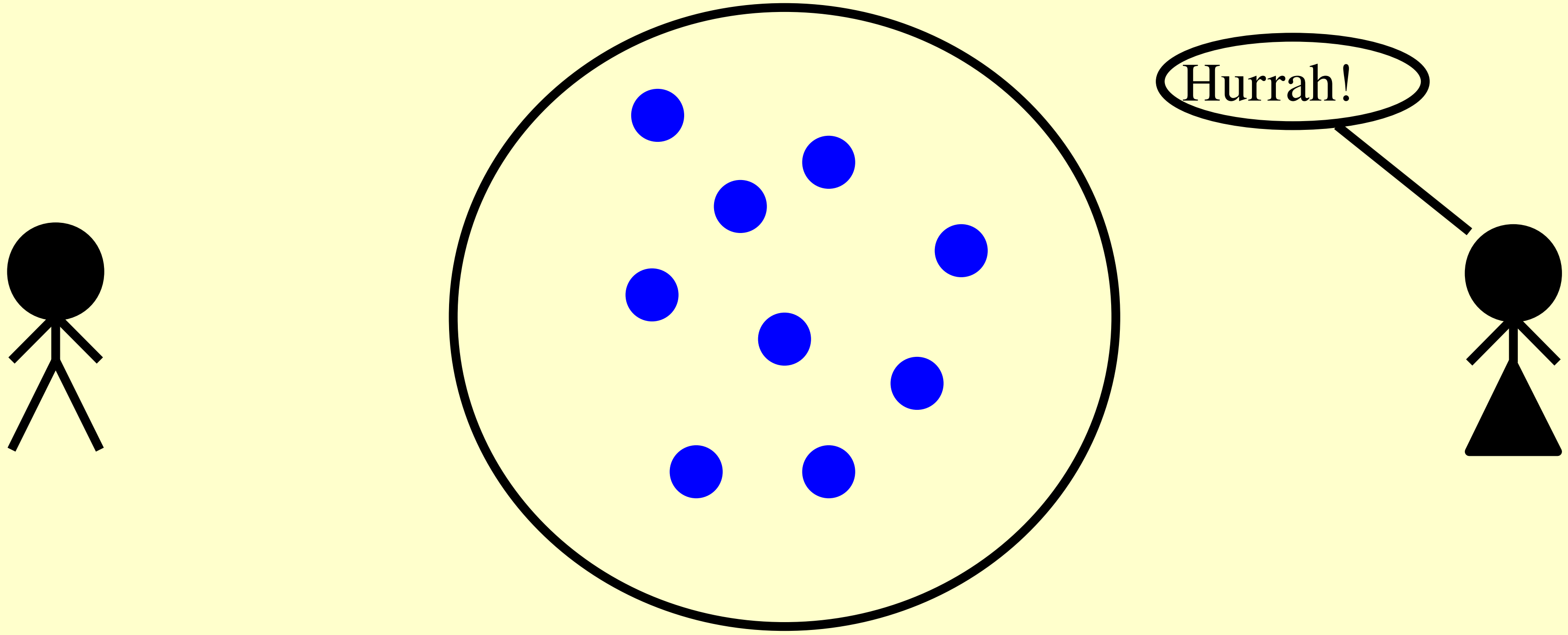
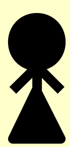


Eloise can wins with high probability: almost true

Probabilistic Dependence Logic

Let us adapt Hodges' semantics to formalize this:

Probabilistic Teams = probability distributions $\boldsymbol{\mu}$ over assignments (= prob. distr. of partial plays reaching some ψ, FV(ψ) $\subseteq$ Dom(**X**));

$\mathbf{V}_{\mu}(\psi) \geq$ r iff  has a uniform behavioral strategy which guarantees a payoff of at least *r* under $\boldsymbol{\mu}$;

Supplementation: $\boldsymbol{\mu}[\mathbf{F}/x](s[m/x]) = \boldsymbol{\mu}(s) \cdot \mathbf{F}(s)(m)$;

Linear decomp.: $\boldsymbol{\mu} = p\, \boldsymbol{\xi}_1 + (1\text{-}p)\, \boldsymbol{\xi}_2$, $p \in [0..1]$.

Probabilistic Dependence Logic

- If ϕ literal, $\mathbf{V}_\mu(\psi) = \sum_{s \models \phi} \mu(s)$;
- $\mathbf{V}_\mu(\phi \vee \psi) = \sup \{p\, \mathbf{V}_{\xi_1}(\phi) + (1-p)\mathbf{V}_{\xi_2}(\psi) :$
  $p\, \boldsymbol{\xi}_1 + (1-p)\, \boldsymbol{\xi}_2 = \boldsymbol{\mu}\}$;
- $\mathbf{V}_\mu(\phi \wedge \psi) = \inf \{p\, \mathbf{V}_{\xi_1}(\phi) + (1-p)\mathbf{V}_{\xi_2}(\psi) :$
  $p\, \boldsymbol{\xi}_1 + (1-p)\, \boldsymbol{\xi}_2 = \boldsymbol{\mu}\}$;
- $\mathbf{V}_\mu((\exists x \backslash W)\ \psi) = \sup \{\mathbf{V}_{\mu\,[F/x]}(\psi) :$
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Probabilistic Dependence Logic

- $\mathbf{V}_\mu((\forall x \setminus W)\ \psi) = \inf \{\mathbf{V}_{\mu\ [F/x]}(\psi) :$
  $s \equiv_W s' \Rightarrow F(s) = F(s')\}.$

If the game for ϕ admits a behavioral equilibrium (for example, if the game is of perfect recall) then the value of the game is $V_{\{\emptyset\}}(\phi)$, and moreover

$$V_{\{\emptyset\}}(\neg\phi) = 1 - V_{\{\emptyset\}}(\phi).$$

Probabilistic Dependence Logic

In general, these are games of imperfect recall, so sometimes no behavioral equilibrium exists.

Consider: $M=\{0,1\}$, $\phi := \exists x(\exists y \backslash \{\ \})(\forall z \backslash \{\ \})(x=y \wedge x \neq z)$

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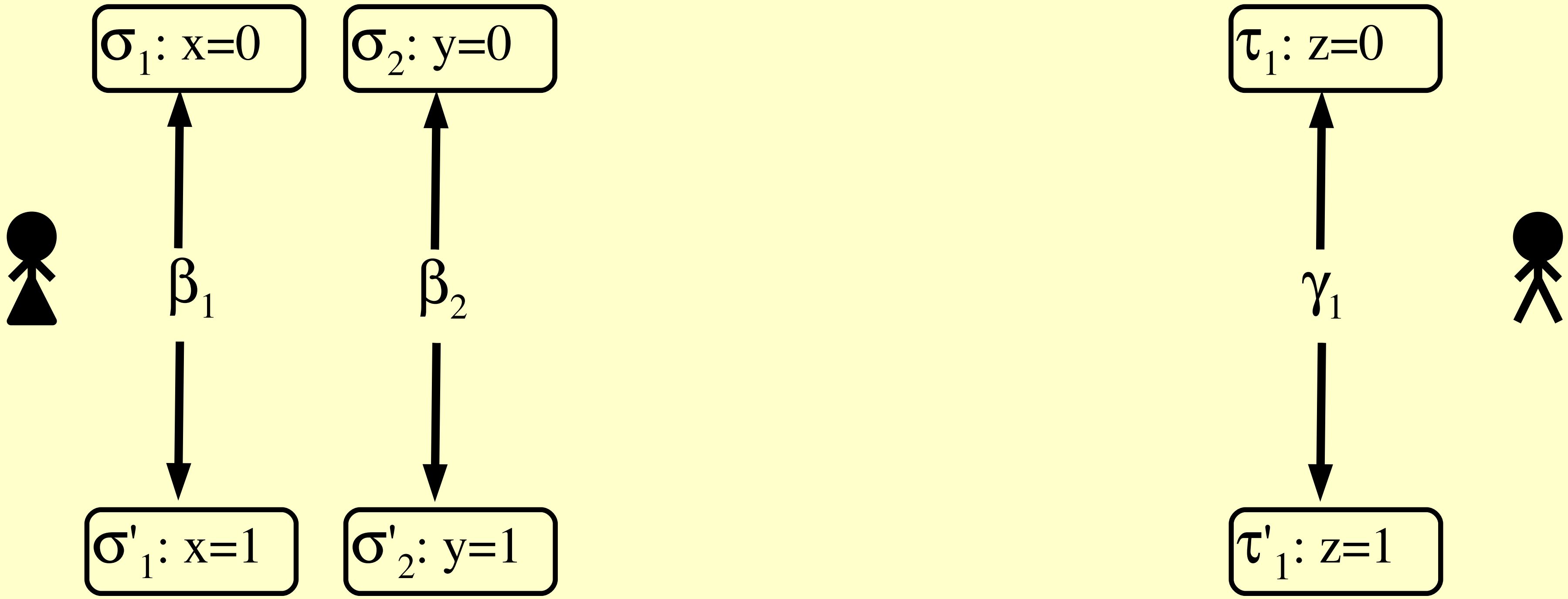

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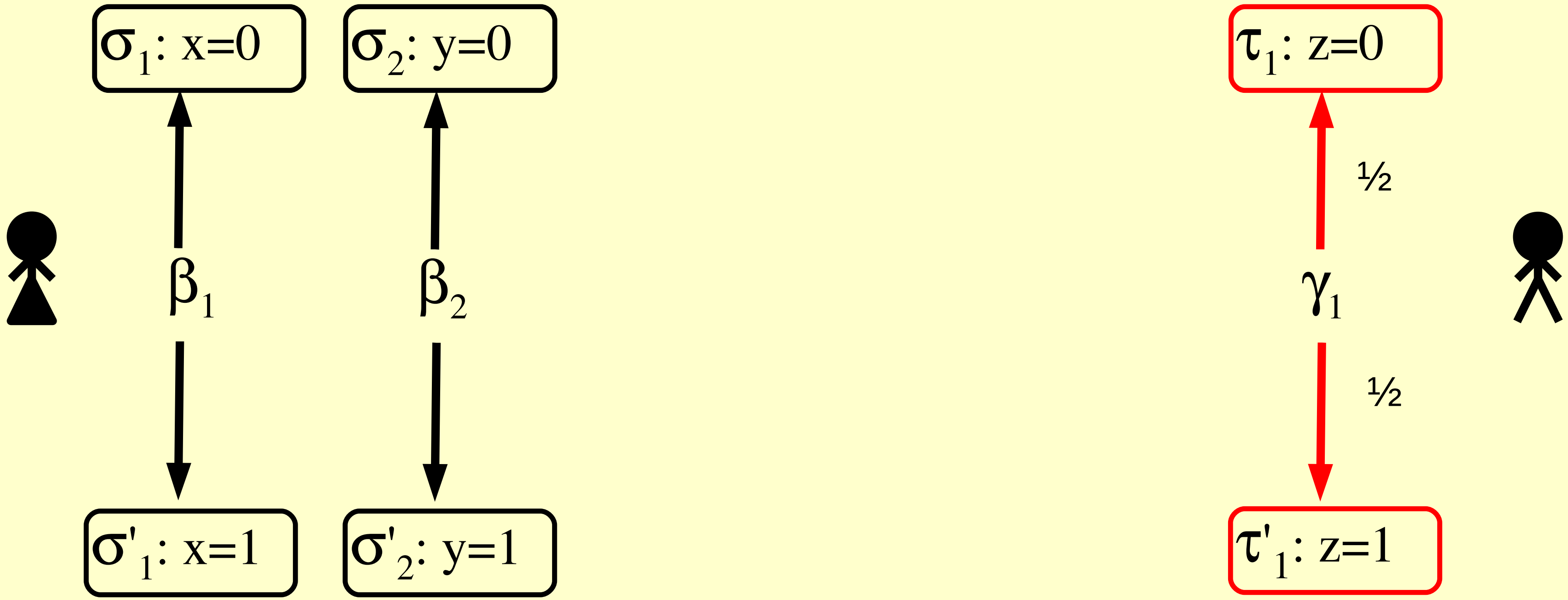

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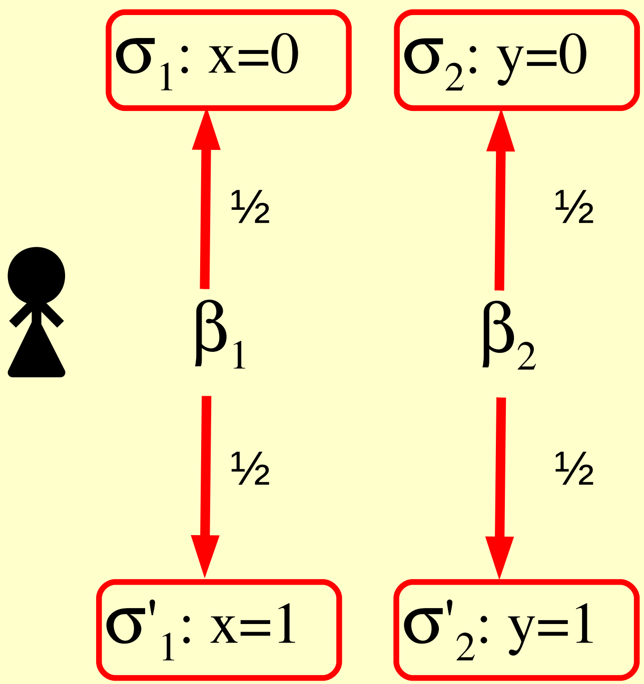

1/8: x=0, y=0, z=0
1/8: x=0, y=0, z=1
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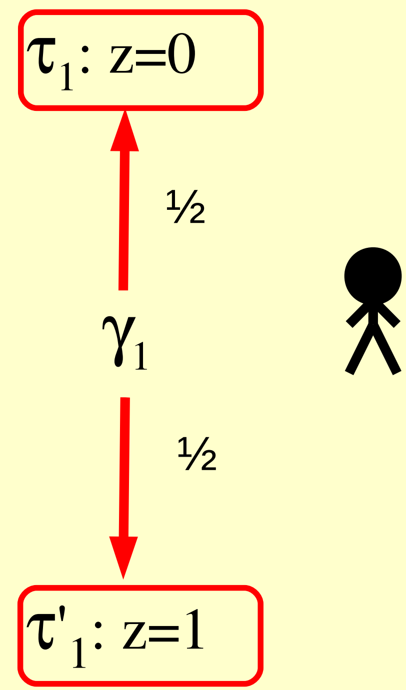
Payoff = 1/4



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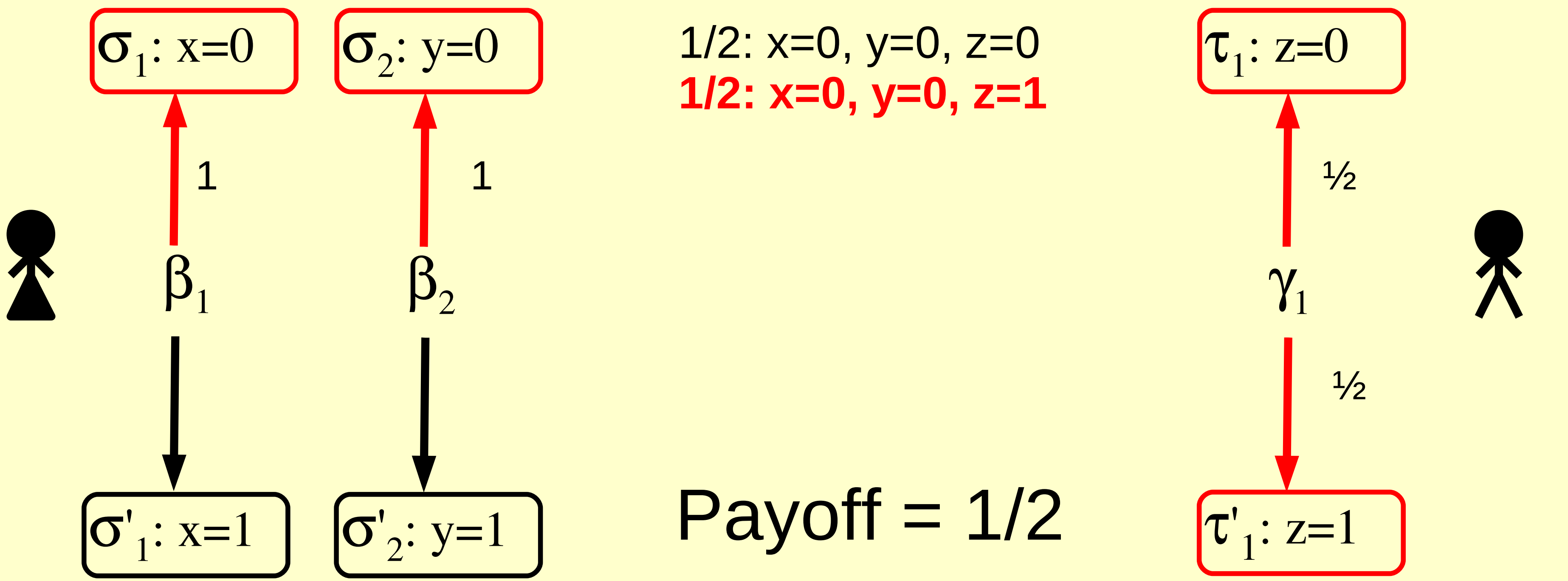

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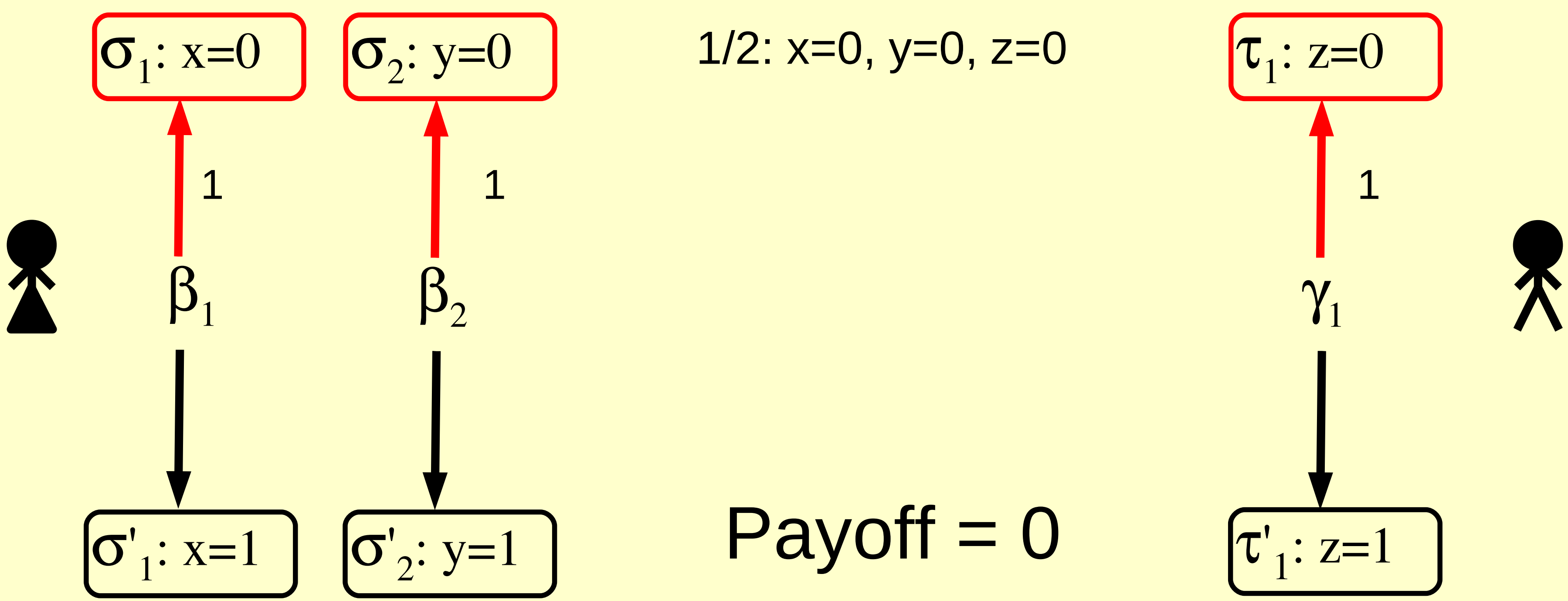

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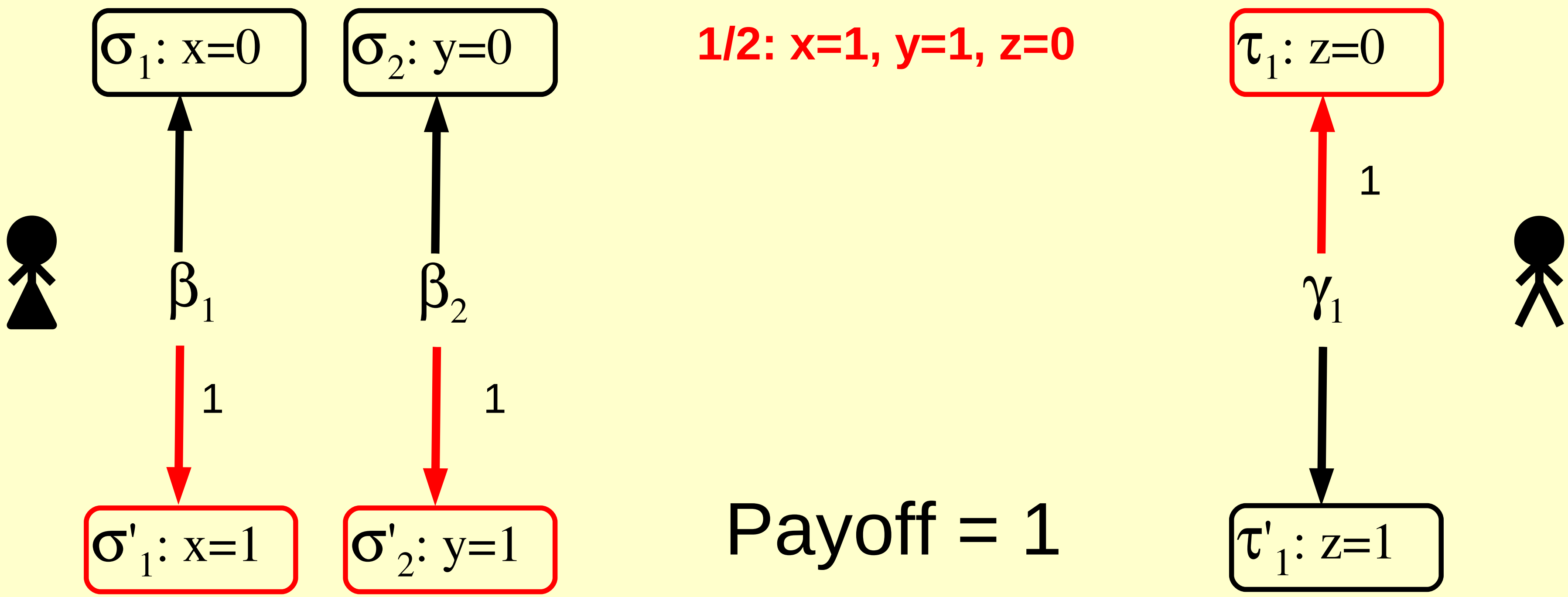

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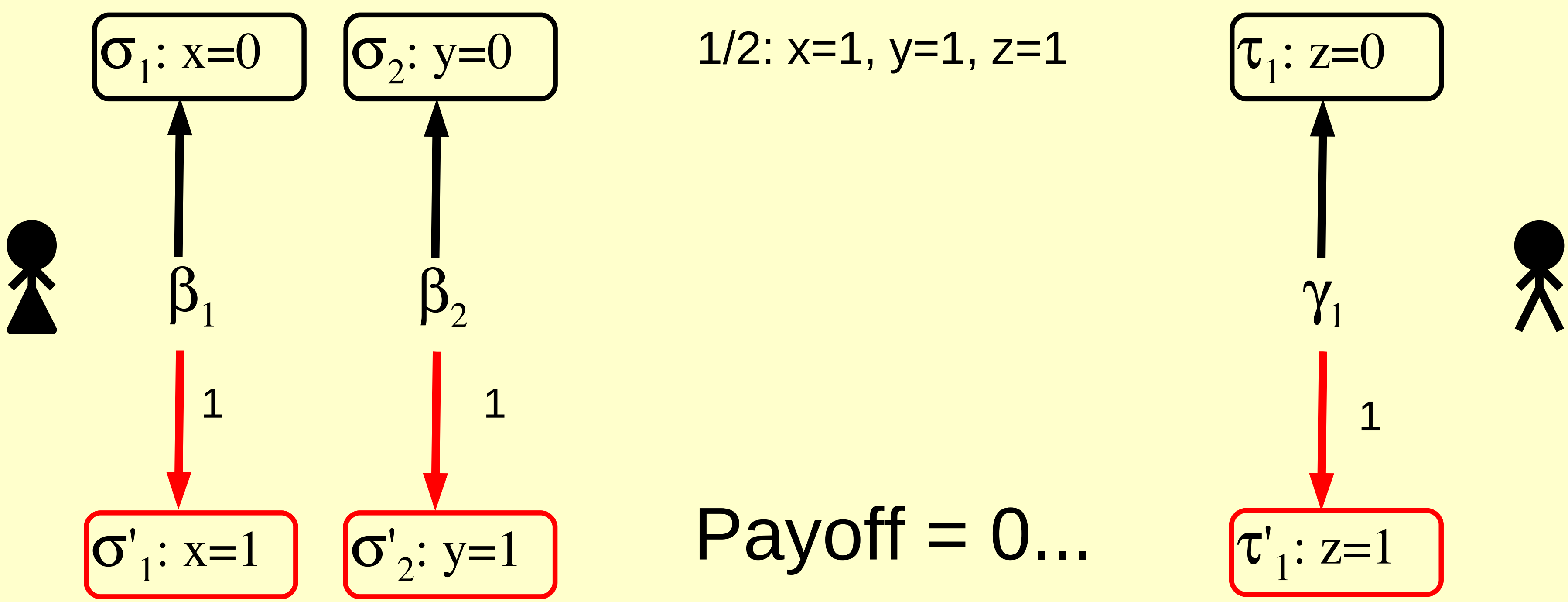

Probabilistic Dependence Logic

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Consider: M={0,1}, $\phi := \exists x (\exists y \backslash \{\ \})(\forall z \backslash \{\ \})(x=y \wedge x \neq z)$

The (mixed strategy) value of the game is ½, for
{(½: x=y=0, ½: x=y=1); (½: z=0, ½: z=1)}

But when using behavioral strategies Eloise cannot coordinate her first and second moves in this way.

Probabilistic Dependence Logic

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Using the compositional semantics,

$V(\exists x (\exists y \backslash \{\,\})(\forall z \backslash \{\,\})(x=y \wedge x \neq z)) = 1/4$;

Probabilistic Dependence Logic

In general, these are games of imperfect recall, so sometimes no behavioral equilibrium exists.

Consider: M={0,1}, $\phi := \exists x (\exists y \backslash \{\}) (\forall z \backslash \{\}) (x = y \wedge x \neq z)$

Using the compositional semantics,

$V(\exists x (\exists y \backslash \{\}) (\forall z \backslash \{\}) (x = y \wedge x \neq z)) = ¼$;

$V(\forall z (\exists x \backslash \{\}) (\exists y \backslash \{\}) (x = y \wedge x \neq z)) = ½$!

Probabilistic Dependence Logic

Two possible answers:

1) Find the values of the mixed equilibria, either by adding quantifiers before the formula to encode the mixed strategies (Galliani 2008, submitted) or modifying the rules of the compositional semantics (Mann and Galliani, in preparation).

2) Consider the values computed to the semantics in themselves, even if no equilibrium (cfr. upper and lower values, etc...). Multiplayer (Cfr. Abramsky 2006)